A preliminary set of methods and tools for computing or approximating net equilibria, net flows and the core, price of anarchy, for restricted agent rationality
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Work Package 4.1: Game theoretic approaches to cooperation and competition in dynamic large nets

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1 Introduction

Most of the existing, such as the Internet, and foreseen complex networks are operated and built by thousands of large and small entities (autonomous agents), collaborating with admirable effectiveness to process and deliver end-to-end flows originating and terminating in any of them. The distributed nature of the Internet has as a consequence a lack of coordination among its users. Instead, each user attempts to obtain maximum performance according to his own parameters.

Methods from Game Theory and Mechanism Design are proved to be a powerful mathematical tool in order to understand, control and efficiently design such dynamic, complex networks. Game theory can provide a good starting point for Computer Scientists in order to understand selfish rational behavior of complex nets of many agents (players). Such a scenario is readily modeled using Game Theory techniques in which players with potentially different goals participate under a common setting with well prescribed interactions (in this case, TCP/IP protocols).

Nash Equilibrium [Nas50, Nas51] stands out as the predominant concept of rationality in non-cooperative settings. Thus, Game Theory and its notions of equilibria provide a rich framework for modelling the behavior of selfish agents in these kinds of distributed or networked environments and offer mechanisms to achieve efficient and desirable global outcomes in spite of the selfish behavior.

Mechanism Design asks how one can design systems so that agents’ selfish behavior results in desired system-wide goals. Algorithmic Mechanism Design additionally considers computational tractability to the set of concerns. Work in Algorithmic Mechanism Design classically focuses on the complexity of centralized implementations of game-theoretic mechanisms for distributed optimization problems. Moreover, in such a huge network each player does not have access to and may not process complete information. The notion of bounded rationality for network agents and the design of distributed algorithms have been successfully utilized to capture the aspect of incomplete information.

In the context of this research framework, within Subproject “Game Theoretic and Organizational Economics Inspired Approaches” we have been dwelling into a sequence of related fields. Deliverable D4.1.1 [D4104] contained a comprehensive catalog of open problems in Game Theory w.r.t. computational efficiency as recorded in the beginning of the project. In this second deliverable, we present some of the most important advances observed during the last years towards answering some of the quite important open problems indicated in [D4104]. For the fields addressed, not only we have absorbed the existing worldwide knowledge but, for some of them, we have also contributed towards their progress. The areas addressed are the followings:

Congestion Games A central problem arising in the management of large-scale communication networks like the Internet is that of routing traffic through the network. Due to the large size of these networks, however, it is often impossible to employ a centralized traffic management. A natural assumption to justify the absence of central regulation is that network users behave selfishly and aim at optimizing their own individual welfare. One way to address this problem, proved to be much powerful, is to model it as a non-cooperative multi-player game and formalize it as a (un-)weighted congestion game; this is a selfish routing game with splittable or unsplittable flows. In this document, we later provide a set of methods and tools for computing Nash equilibria for such network settings.
Price of Anarchy  We survey precise and approximate computations of the Price of Anarchy; this is the cost of selfish cooperation in dynamic, large-scale nets, compared to centralized hypothetical solutions. We consider the Price of Anarchy for some of the most important network problems, modeled by non-cooperative games; for example, we consider routing and security problems.

Selfish Routing with Incomplete Information  The notion of bounded rationality in networks with incomplete information can be addressed by Bayesian games and also by congestion games with player specific payoff functions. We will survey methods and tools for approximating net equilibria and net flows for restricted agent rationality.

Game Theoretic Analysis of Internet Switching  We survey the problem of Internet switching, where traffic is generated by selfish users concentrated on packetized (TCP-like) traffic models; this is more realistic than the widely used fluid model. We survey a game-theoretic model and analysis of the Internet switching problem.

Mechanism design  Mechanism design is a subfield of Game Theory and Microeconomics which deals with the design of protocols for rational agents. Generally, a Mechanism Design problem can be described as the task of selecting from a collection of feasible games, a game which will yield desirable results for the designer. The routing problem in large-scale networks, where users are instinctively selfish, can be modeled by a non-cooperative game. Such a game could impose strategies that might induce an equilibrium closer to the overall optimum. These strategies are enforced through pricing mechanisms [FCHPSS00], algorithmic mechanisms [NR99] and network design [KLO95, Rou01a].

Stackelberg Games  We will consider network routing games from the network designer’s point of view. In particular, the network administrator or designer can define prices and rules, or even construct the network, in a way that induces near-optimal performance when the users act selfishly to the system. Particularly interesting is the approach where the network manager takes part in the non-cooperative game. The manager has the ability to control centrally a part of the system resources, while the rest resources are used by the selfish users. This approach has been studied through Stackelberg or Leader-Follower games [BO99, KLO97, CDR03]. The advantage of this approach is that it might be easier to be deployed in large-scale networks. This is so since there is no need to add extra components to the network or to exchange information between the users of the network. In a Stackelberg game, one player acts as a leader (here, the centralized authority interested in optimizing system performance) and the rest as followers (here, the selfish users). The problem is then to compute a strategy for the leader (a Stackelberg strategy) that induces the followers to react in a way that (at least approximately) minimizes the total latency in the system. Selfish routing games can be modeled as a Stackelberg game. We will survey issues related to how the manager should assign the flow under his control into the system so as to induce optimal cost incurred by the selfish users.

Pricing mechanisms  Pricing mechanisms in resource allocation problems aim to allocate resources in such a way that those users who derive greater utility from the network are not denied access due to other users who place a lower value on it. In other words, pricing mechanisms are designed to guarantee economic efficiency. We will discuss cost-sharing mechanisms for pricing the competitive usage of a collection of resources by a collection of selfish agents, each coming with an individual demand.
Network Security Games  We will also consider security problems in dynamic, large-scale, distributed networks. This problem can be modeled as a concise, non-cooperative multi-player game played on a graph. We will investigate associated Nash equilibria in such network security games.

The Core  The nucleolus is a solution concept for coalition form games with transferable utility that was first introduced by Schmeidler [Sch69]. Osborne and Rubinstein [OR94] proposed a new but equivalent definition of the nucleolus in terms of objections and counter-objections in negotiations. The nucleolus is a payoff division such that for every objection to it, there is a counter-objection. In recent years, there has been an increased interest in computational complexity aspects of solution concepts in cooperative game theory, such as the nucleolus of a game. Here, we will investigate issues related to the nucleolus of games with an emphasis on flow games.

Complexity of Computing Equilibria  The investigation of the complexity of finding a Nash equilibrium in a general game is definitely a fundamental task for the development of Algorithmic Game Theory. Answers to such questions are expected to have great practical impact on both the analysis of the behavior of antagonistic networks and the network designers directions. Finding a Nash equilibrium in a game with two players could potentially be easier than for many players for several reasons. First, the zero-sum version of the game can be solved in polynomial time by linear programming. Secondly, it admits a polynomial size rational number solution, while games with three or more players may only have solutions in irrational numbers. This reasoning justified the identification of the problem of finding Nash equilibria for a 2-player game as one of the most important open questions in the field of Algorithmic Game Theory. The complexity of this problem was very recently settled in a perhaps surprising way in a series of breakthrough papers. Here, we will survey the worldwide literature related to this problem and the recent progress to it.

The scientific background related to this document can be found in deliverable D4.2.1 [D4204]. That document provides a comprehensive evaluation of fitness of the currently known mechanisms with respect to selfish cooperation in networks.

2 Congestion Games

2.1 The General Framework

2.1.1 Congestion Games

Rosenthal [Ros73] introduced a special class of strategic games, now widely known as congestion games. Here, the strategy set of each player is a subset of the power set of a set of resources. The players share a private objective function, defined as the sum (over their chosen resources) of functions in the number of players sharing this resource. In his seminal work, Rosenthal showed with the help of a potential function that congestion games (in sharp contrast to general strategic games) always admit at least one pure Nash equilibrium. An extension to congestion games are weighted congestion games, in which the players have weights and thus different influence on the congestion of the resources. In (weighted) network congestion games, the strategy sets of the players correspond to paths in a network.
2.1.2 Price of Anarchy

In order to measure the degradation of social welfare due to the selfish behavior of the players, Koutsoupias and Papadimitriou [KCHP99] introduced in their seminar work a global objective function, usually coined as Social Cost. They defined the Price of Anarchy, also called Coordination Ratio, as the worst-case ratio between the value of Social Cost in a Nash equilibrium and that of some Social Optimum. Thus, the Coordination Ratio measures the extent to which non-cooperation approximates cooperation.

As a starting point for studying the Coordination Ratio, Koutsoupias and Papadimitriou considered a very simple weighted network congestion game, now known as KP-model. Here, the network consists of a single source and a single destination (single-commodity network) which are connected by parallel links. The load on a link is the total weight of players assigned to this link. Associated with each link is a capacity representing the rate at which the link processes load. Each of the players selfishly routes from the source to the destination by using a probability distribution over the links. The private objective function of a player is defined as its expected latency. In the KP-model, the Social Cost is defined as the expected maximum latency on a link, where the expectation is taken over all random choices of the players.

Mavronicolas and Spirakis [MS01] introduced the notion of a fully mixed Nash equilibrium, in which each player chooses every link with positive probability. Gairing et al. [GTLM+03] conjectured that, in case of its existence, the fully mixed Nash equilibrium is the worst Nash equilibrium with respect to Social Cost. This so-called Fully Mixed Nash Equilibrium Conjecture is simultaneously intuitive and significant. It is intuitive because the fully mixed Nash equilibrium favors collisions between different players (since each player assigns its item with positive probability to every link). This increased probability of collisions should favor an increase to Social Cost. The conjecture is also significant since it identifies the worst-case Nash equilibrium over all instances.

Recently, the KP-model was extended to restricted strategy sets [GTLMM04a], where the strategy set of each player is a subset of the links. Furthermore, the KP-model was extended to general latency functions and studied with respect to different definitions of Social Cost [AAE05]. Inspired by the arisen interest in the Price of Anarchy, the much older Wardrop-model was reinvestigated in [RT02]. In this weighted network congestion game, weights can be split into arbitrary pieces. The social welfare of the system is defined as the sum of the edge latencies (Sum or Total Social Cost). An equilibrium in the Wardrop model can be interpreted as a Nash equilibrium in a game with infinitely many players, each carrying an infinitesimal amount of weight.

In [KCHP99], the authors consider the objective of (expected) maximum latency (also called Maximum Social Cost) for a weighted congestion game in uniformly related parallel links. The Price of Anarchy for this game is shown to be $\Theta\left(\frac{\log m}{\log \log m}\right)$ if either the users or the links are identical [CV02, KMS03] and $\Theta\left(\frac{\log m}{\log \log \log m}\right)$ for weighted users and uniformly related links [CV02]. On the other hand, [CKV02] shows that the Price of Anarchy is far worse and can be even unbounded for arbitrary latency functions. For uniformly related parallel links, identical users, and the objective of total latency, the Price of Anarchy is $1 - o(1)$ for the general case of mixed equilibria and $4/3$ for pure equilibria [LMMR04, GTLM+04]. For identical users and polynomial latency functions of degree $d$, the Price of Anarchy is $d^{\Theta(d)}$ [GTLMM04b, GTLM+04].

Christodoulou and Koutsoupias in [CK05] consider the Price of Anarchy of pure Nash equilibria in congestion games with linear latency functions. They showed that for general (asymmetric) games, the Price of Anarchy of Maximum Social Cost is $\Theta(\sqrt{N})$, where $N$
is the number of players. For all other cases of symmetric or asymmetric games and for both Maximum and Average Social Cost, the Price of Anarchy is shown to be $5/2$.

### 2.2 Algorithms and Complexity

A comprehensive survey of some of the most important recent advances in the literature for atomic congestion games is contained in [KS05]. That work is an overview of the extensive expertise on (mainly network) congestion games and the closely related potential games, that has been developed in various disciplines (e.g., Economics, Computer Science and Operations Research) under a common formalization and modeling. In particular, the survey is not only an exposition of known results, but goes deep into the details of some of the most characteristic results in the area in order to compile a useful toolbox that Game Theory provides in order to study antagonistic behavior due to congestion phenomena in real-life problems.

#### 2.2.1 Selfish Unsplittable Flows

In [FKS05a], Fotakis et al. study congestion games in which selfish users with varying service demands for the system resources may request a joint service from arbitrary subsets of resources. Each user’s demand has to be served unsplittably from a specific subset of resources. In that work, it is proved that the weighted congestion games are no longer isomorphic to the well known potential games, although this was true for the case of users with identical service demands. The authors also demonstrate the power of the network structure, when users of varying demands request service. For very simple networks, they show that there may not exist a pure Nash equilibria, which is not true for the case of parallel links network or for the case of infinitely splittable service demands. Furthermore, the authors propose a family of networks (called layered networks) for which they show the existence of at least one pure Nash equilibria when each resource (that is, link) charges its users with delay identical to its load. Finally, the same work considers the Price of Anarchy in the family of layered networks in the case when each resource delay equals its load. It is shown that the Price of Anarchy for this case is $\Theta\left(\frac{\log m}{\log \log m}\right)$. That is, within constant factors, the worst network is the simplest one (the parallel links network). This implies that in this family of networks, the network structure does not essentially affect the quality of the congestion games played on the network.

[PS05] considers selfish routing in single-commodity networks, where selfish users select paths to route their loads (represented by arbitrary integer weights). It considers identical delay functions for the links of the network. That work focuses also on the algorithm suggested in [FKS05a]; this is a potential-based method for finding pure Nash equilibria in such networks. The analysis of this algorithm in [FKS05a] gives an upper bound on its running time, which is polynomial in $n$ (the number of users) and the sum $W$ of their weights. This bound can be exponential in $n$ when some weights are superpolynomial. Therefore, the algorithm was indeed proved to be pseudo-polynomial. [PS05] provides strong experimental evidence that this algorithm actually converges to a pure Nash equilibria in strong polynomial time in $n$ (independent of the weights values). In addition, the authors propose an initial allocation of users to paths that dramatically accelerates this algorithm, as opposed to an arbitrary initial allocation. A byproduct of that work is the discovery of a weighted potential function when link loads are exponential to their loads. This guarantees the existence of pure Nash equilibria for these delay functions and extends the result of [FKS05a].
2.2.2 Worst-Case Equilibria

In [FV05b], S. Fischer and B. Vöcking studied the selfish routing game associated with the KP-model [KCHP99], where \( n \) weighted jobs are allocated to \( m \) identical machines. Gairing et al. [GTLM+03] had conjectured that the fully mixed Nash equilibrium is the worst Nash equilibrium for this game with respect to the expected maximum load over all machines. The known algorithms for approximating the Price of Anarchy relied on proved cases of that conjecture. In [FV05b], the authors interestingly present a counterexample to the conjecture showing that fully mixed equilibria cannot be generally used to approximate the Price of Anarchy within reasonable factors. In addition, they present an algorithm that constructs the so-called concentrated equilibria that approximate the worst-case Nash equilibrium within constant factors.

2.2.3 Symmetric Congestion Games

[FKS05b] continues the work of [FKS05a] and study computational and coordination issues of Nash equilibria in symmetric network congestion games. A game is symmetric if all users have the same strategy set and users costs are given by identical symmetric functions of other users’ strategies. In congestions games, the users are identical, so that a common strategy set implies symmetry. This work proposed a simple and natural greedy method (which is called the Greedy Best Response – GBR), that computes a pure Nash equilibria. In this algorithm, each user plays only once and allocates his traffic to a path selected via a shortest path computation. Then, it is shown that this algorithm works for series-parallel networks, when users are identical, or when users are of varying demands but have the same best response strategy for any initial network traffic (this is called the Common Best Response property). The authors also give constructions where the algorithm fails if either the above condition is violated (even for series-parallel networks), or the network is not series-parallel (even for identical users). Thus, they essentially indicate the limits of the applicability of this greedy approach.

The same work [FKS05b] studies also the Price of Anarchy for the objective of maximum latency. It is proved that for any network of \( m \) uniformly related links and for identical users, the Price of Anarchy is \( \Theta \left( \frac{\log n}{\log \log m} \right) \). This result is complementary (and somewhat orthogonal) to a similar result provided in [FKS05a] for the case of players of varying weights to be routed in a layered network.

2.2.4 Exact Price of Anarchy

Exact bounds on the Price of Anarchy or unweighted and weighted congestion games with polynomial latency functions are provided in [ADGaFS06]. The authors use the total latency as the Social Cost measure. This improves on results by Awerbuch et al. [AAE05] and Christodoulou and Koutsoupias [CK05], where non-matching upper and lower bounds were are given.

For the case of unweighted congestion games, in the same work [ADGaFS06] it is shown that the price of anarchy (PoA) is exactly

\[
\text{PoA} = \frac{(k + 1)^{2d+1} - k^{d+1}(k + 2)^d}{(k + 1)^{d+1} - (k + 2)^d + (k + 1)^d - k^{d+1}},
\]

where \( k = |\Phi_d| \) and \( \Phi_d \) is a natural generalization of the golden ratio to larger dimensions such that \( \Phi_d \) is the solution to \( (\Phi_d + 1)^d = \Phi_d^{d+1} \). Prior to that paper, the best known upper and lower bounds were shown to be of the form \( d^{d(1-o(1))} \) [CK05]. However, the term \( o(1) \) still hide a significant gap between the upper and the lower bound.
For weighted congestion games, the authors show that the Price of Anarchy (PoA) is exactly
\[ \text{PoA} = \Phi d + 1. \]
This result closes the gap between the so far best upper and lower bounds of \( O(2^d d^{d+1}) \) and \( \Omega(d^{d/2}) \) from [AAE05].

The authors show that the above values on the Price of Anarchy also hold for the subclasses of unweighted and weighted network congestion games. For the upper bounds, the authors use a similar analysis as in [CK05]. The core of their analysis is to determine parameters \( c_1 \) and \( c_2 \) such that
\[ y \cdot f(x + 1) \leq c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y) \]
for all polynomial latency functions of maximum degree \( d \) and for all reals \( 1 \leq x, y \leq 0 \).
For the case of unweighted demands, it suffices to show (1) for all pairs of integers \( x \) and \( y \). In order to prove their upper bound, Christodoulou and Koutsoupias [CK05] looked at (1) with \( c_1 = \frac{1}{2} \) and gave an asymptotic estimate for \( c_2 \). In the analysis presented in [ADGaFS06] both parameters \( c_1 \) and \( c_2 \) are optimized. This optimization process required new mathematical ideas and is non-trivial.

3 Selfish Routing with Incomplete Information

In his seminal work, Harsanyi [Har67] introduced an elegant approach to study non-cooperative games with incomplete information, where the players are uncertain about some parameters. To model such games, he introduced the Harsanyi transformation, which converts a game with incomplete information to a strategic game where players may have different types. In the resulting Bayesian game, the players’ uncertainty about each other’s types is described by a probability distribution over all possible type profiles.

The problem of selfish routing with incomplete information has recently been faced via the introduction of new suitable models and the development of new methodologies that help to analyze such network settings. In particular, new selfish routing games with incomplete information have been introduced, called Bayesian routing games [GaKT05]. Furthermore, the same problem can be viewed as a congestion game where latency functions are player-specific [GMT05], or a congestion game under the restriction that the link for each user must be chosen from a certain set of allowed links for the user [EGTL+05]. A relevant, also important, problem is that of adaptive routing in networks by selfish users that lack central control. This important problem has not yet been addressed.

3.1 Bayesian Routing Games

In [GaKT05], a particular selfish routing game with incomplete information, called Bayesian routing game is introduced. Here, \( n \) selfish users wish to assign their traffic to one of \( m \) parallel links. Users do not know each other’s traffic. Following Harsanyi’s approach, the authors introduce for each user a set of possible types. This work contains a comprehensive collection of results for this Bayesian routing game.

In [GaKT05], with respect to the problem of the existence and computational complexity of pure Bayesian Nash equilibria, it is proved with the help of a potential function, that every Bayesian routing game has a pure Bayesian Nash equilibrium. This result can also be generalized to a larger class of games, called weighted Bayesian congestion games. For the case of identical links and independent type distributions, it is shown that a pure Bayesian Nash equilibrium can be computed in polynomial time. (A probability distribution over all
possible type profiles is independent if it can be expressed as the product of independent probability distributions, one for each type.)

In the same work the authors study structural properties of fully mixed Bayesian Nash equilibria for the case of identical links and show that they maximize Individual Cost. In general, there exists more than one fully mixed Bayesian Nash equilibrium. The authors provide a characterization of the class of fully mixed Bayesian Nash equilibria for the case of independent type distribution; the characterization determines in turn the dimension of the space of fully mixed Nash equilibria.

Finally, then authors in [GaKT05] consider the Price of Anarchy for the model of identical links for three Social Cost measures; that is, Social Cost as expected maximum congestion, Sum of Individual Costs and Maximum Individual Cost. For the latter two, (asymptotic) tight bounds using the proven structural properties of fully mixed Bayesian Nash equilibria were provided.

### 3.2 Player-Specific Latency Functions

[GMT05] studies (weighted) network congestion games under the aspect of incomplete knowledge also about the system. As shown there, such Bayesian routing games can be transformed into routing games where the latency functions are player-specific. There, $n$ selfish players wish to route their traffic through a shared network. This work considers both the case of splittable and unsplittable traffics.

In this perspective, the proposed models generalize the two famous models of selfish routing, namely weighted (network) congestion games and the Wardrop model to accommodate player-specific latency functions. Latency functions may be arbitrary non-decreasing functions; however, many of their results assume that the latency function for player $i$ on resource $j$ is a linear function $f_{ij}(x) = a_{ij}x + b_{ij}$ where $a_{ij} \geq 0$ and $b_{ij} \geq 0$. They use the term player-specific capacities to denote a game where $b_{ij} = 0$ in all latency functions.

They derive several interesting results on the existence and computational complexity of (pure) Nash equilibria and the Price of Anarchy. For routing games on parallel links with player-specific capacities, they introduce two new (potential) functions, one for unsplittable and one for splittable traffics. The first potential function is used to prove, for the case of unsplittable traffics, that games with unweighted players possess the finite improvement property. It is also shown that games with weighted players do not possess the finite improvement property in general, even if $n = 3$. The second function is a convex function that plays the role of a potential function for the case of splittable traffics. This convex function is minimized if and only if the corresponding assignment is a Nash equilibrium. This result implies that a Nash equilibrium can be computed in polynomial time.

The same work proves upper and lower bounds on the Price of Anarchy under a certain restriction on the linear latency functions. For the case of unsplittable traffics the upper and lower bounds are asymptotically tight. All results on the Price of Anarchy translate to general congestion games.

### 3.3 Network Uncertainty in Selfish Routing

The problem of selfish routing in the presence of incomplete network information is also studied in [GTPP06]. This work proposes a new model for selfish routing in the presence of incomplete network information. The proposed model captures situations where the users have incomplete information regarding the link capacities. Such uncertainty may be caused if the network links are complex paths created by routers which are constructed differently on separate occasions and according to the presence of congestion or link failures.
The new, extremely interesting model presented in [GTPP06] consists of a number of users who wish to route their traffic on a network of \( m \) parallel links with the objective of minimizing their latency. In order to capture the lack of precise information on the capacity of the network links, it is assumed that links may present a number of different capacities. Each user’s uncertainty about the capacity of the links is modeled via a probability distribution over all possibilities. Furthermore, it is assumed that users may have different sources of information regarding the network and, therefore, take their probability distributions to be distinct from each another. This gives rise to a model with user-specific payoff functions, where each user uses its distinct probability distribution to take decisions as to how to route its traffic.

The authors propose polynomial-time algorithms for computing some special cases of pure Nash equilibria and demonstrate that the counter-example presented in [Mil96], showing that pure Nash equilibria may not exist in the general case, does not apply in their model. Thus, they identify an interesting open problem in this area, that of existence of pure Nash equilibria in the general case. Also, two different expressions for the Social Cost and the associated Price of Anarchy are identified and employed. For the latter, the authors obtain upper bounds in the general case and improved upper bounds for several special cases.

In the same work, the authors show how to compute the fully mixed Nash equilibrium, and show that when it exists it is unique. Also, it is shown that for certain instances of the game, fully mixed Nash equilibria assign all links to all users equiprobably. Finally, it verifies the Fully-Mixed Nash Equilibrium conjecture, by proving that the fully mixed Nash equilibrium maximizes the social welfare.

### 3.4 Restricted Selfish Scheduling

The paper [EGTL+05] considers selfish routing problems in networks under the restriction that the link for each user must be chosen from a certain set of allowed links for the user. It is assumed that each user has access (that is, finite cost) to only two machines; its cost on other machines is infinitely large, giving it no incentive to switch there. The (expected) cost of a user is the (expected) load of the machine it chooses. Interaction with just a few neighbors is a basic design principle to guarantee efficient use of resources in a distributed system. Restricting the number of interacting neighbors to just two is then a natural starting point for the theoretical study of the impact of selfish behavior in a distributed system with local interactions.

In that work, a simple, graph-theoretic model for selfish scheduling among \( m \) non-cooperative users over a collection of \( n \) machines was introduced and studied. There, each user is restricted to assign its unsplittable load to one from a pair of machines that are allowed for the user. The authors model these bounded interactions using an interaction graph, whose vertices and edges are the machines and the users, respectively. They also study the impact of the modeling assumptions on the properties of Nash equilibria in their new model.

In that same work, it is proved that the parallel links graph is the best-case interaction graph – the one that minimizes expected makespan of the standard fully mixed Nash equilibrium – among all 3-regular interaction graphs. The proof employs a graph-theoretic lemma about orientations in 3-regular graphs, which may be of independent interest.

A lower bound on Coordination Ratio [KCHP99] is also provided. In particular, it is proved that there is an interaction graph incurring Coordination Ratio \( \Omega\left(\frac{\log n}{\log \log n}\right) \). This bound is shown for pure Nash equilibria. Finally, the authors present counterexample interaction graphs to prove that a fully mixed Nash equilibrium may sometimes not exist.
Moreover, they prove existence and uniqueness properties of the fully mixed Nash equilibrium for complete bipartite graphs and hypercube graphs.

3.5 Adaptive Routing with Stale Information

The work [FV05a] considers the problem of adaptive routing in networks by selfish users that lack central control. The main focus of this work is on simple adaption policies, or dynamics, that make use of possibly stale load information. The analysis provided covers a wide class of dynamics encompassing the well-known replicator dynamics and other dynamics known from Evolutionary Game Theory. As a well known problem, it has been often pointed out, that always choosing the best option based on out-of-date information can lead to undesirable oscillation effects and poor overall performance.

In that work, it is shown that it is possible to cope with this problem, and guarantee convergence towards an equilibrium state, for all of this broad class of dynamics, if the function describing the cost of an edge depending on its load is not too steep. Therefore, it turns out that whether or not convergence can be guaranteed depends solely on the size of a single parameter describing the greediness of the agents.

While the best response dynamics, corresponding to always choosing the best option, performs well if information is always up-to-date, it is clear from the results in [FV05a] that this policy fails when information is stale. The authors present a dynamics which approaches the global optimal solution in networks of parallel links with linear latency functions as fast as the best response dynamics does, but which does not suffer from poor performance when information is out-of-date.

4 Game-Theoretic Analysis of Internet Switching

If all Internet users voluntarily deploy a congestion-responsive transport protocol (e.g. TCP [Jac88]), one can design this protocol so that the resulting network would achieve certain performance goals such as high utilization or low delay. However, with the fast growth of the Internet users population, the assumption about cooperative behavior may not remain valid, and in fact it is not. Users are likely to behave selfishly, that is, each user makes decisions so as to optimize its own utility, without coordination with the other users. Buffer sharing and bandwidth allocation problems are prime candidates for investigating the effect of such a selfish behavior.

If a user does not reduce its sending rate upon congestion detection, it can get a better share of the network bandwidth. On the other hand, all users suffer during congestion collapse, since the network delay and the packet loss increase drastically. Therefore, it is important to understand the nature of congestion resulting from selfish behavior. A natural framework to analyze this class of problems is that of non-cooperative games, and an appropriate solution concept is that of Nash equilibrium [Nas51]. Strategies of the users are at a Nash equilibrium if no user can gain by unilaterally deviating from its current policy.

4.1 A Game-Theoretic Model

The problem of Internet switching, where traffic is generated by selfish users, has been considered in [KSLB05]. That work concentrates on a packetized (TCP-like) traffic model which is more realistic than the widely used fluid model. There, it is assumed that routers have First-In-First-Out (FIFO) buffers of bounded capacity managed by the drop-tail policy. The utility of each user depends on its transmission rate and the congestion level.
Since selfish users try to maximize their own utility and disregard the system objectives, Nash equilibria that correspond to a steady state of the system are studied.

[KSLB05] presents a game-theoretic model, called switching model, and an analysis of the Internet switching problem, assuming a FIFO buffer shared by many users. The users compete for the buffer share in order to maximize their throughput, but they suffer from the created congestion. It is assumed that each user knows its buffer usage, the queue length and the buffer size. The goal of a user is to maximize its utility. It is assumed that the utility function of a user increases when its throughput increases and decreases when the network congestion increases.

The model assumes that there are $n$ users and the buffer capacity is $B$, where $B \gg n$. In the model presented, all packets have a constant size. Time is slotted. Every time step each user may or may not send a packet to the buffer. These packets arrive in an arbitrary order and are processed to the buffer management policy one by one. Then, the first packet in the FIFO order is transmitted on the output link.

The drop policy has to decide at each time step which of the packets to drop and which to accept. It can also preempt (drop) accepted packets from the buffer. Under the drop-tail policy, all arriving packets are accepted if the buffer is not full and dropped otherwise (when the buffer is full).

Here, $w_t^i$ denotes the number of packets of user $i$ in the buffer at the beginning of time step $t$. It is called the buffer usage of user $i$. $W_t = \sum_i w_t^i$ denotes the queue length at time $t$. $W_t^i$ denotes the buffer usage of all users but user $i$, i.e., $W_t^i = W_t - w_t^i$. It is assumed that the users are greedy, that is, i.e., they always have data to send and the queue is never empty. $r_t^i = w_t^i/W_t$ denotes the instant transmission rate of user $i$ at time $t$. We say that the system is in a steady state if the queue length remains constant. Intuitively, when a packet of user $i$ is transmitted at time step $t$, at time step $t+1$, user $i$ will send a new packet to the queue. In a steady state, the average transmission rate of user $i$ is $r_i = w_i/W$, where $w_i$ is the average buffer usage of user $i$ and $W$ is the average queue length. Note that in this model, transmission rate is essentially equal to the throughput of the user.

A utility function, which falls into a general family of utility functions increasing in the sending rate and decreasing in the network congestion (see [DM87, She95]), is then defined. However, the rate and the delay (congestion) are rather incomparable values and thus it is natural to normalize one of these parameters. [KSLB05] introduces a novel notion of congestion level, that is the congestion level at time $t$ is $L_t = W_t/B$. Note that the congestion level is zero when the buffer is empty and one when the buffer is full.

The utility of user $i$ at time $t$ is $u_i(w_t^i, W_t^i) = r_t^i \cdot (1 - L_t)$; so, it is proportional to the throughput while decreasing in the congestion level. It is assumed that user $i$ sends a packet to the buffer at time $t$ if its utility increases (that is, $u_i(w_t^i + 1, W_t^i) > u_i(w_t^i, W_t^i)$). Note that users maximize their instantaneous utility, and not the long-term one adapting to the current network conditions. In a steady state, the utility of user $i$ is $u_i(w_i, W_{-i}) = r_i \cdot (1 - L)$, where $L = W/B$. Observe that when the buffer is almost empty, the utility of each user approximately equals its throughput. On the other hand, when the buffer is nearly full, all users have utility close to zero. The latter situation can be viewed as congestion collapse. The strategy of each user is its buffer usage while the strategies of the other users define the background buffer backlog.

The total utility of the users in a Nash equilibrium is $\sum_{i=1}^n u_i(w_i, W_{-i})$. Observe that under an optimal fair centralized policy, all users have equal sending rates of $1/n$ and experience zero delay, which results to a total utility of $1$.

For example, users may send one packet in turn every time step and in this case the link would be fully utilized with zero congestion level. The Price of Anarchy in a Nash
The equilibrium is defined to be $1/\sum_{i=1}^n u_i(w_i, W_{-i})$. The fairness of a Nash equilibrium is also of concern. A Nash equilibrium is said to be fair if all users have the same buffer usage. A Nash equilibrium in a networking environment is interesting only if it can be reached efficiently (in polynomial time). The convergence time to a Nash equilibrium is defined to be the maximum number of time steps required to reach a Nash equilibrium starting from an arbitrary state of the buffer.

In [KSLB05], it is demonstrated that the drop-tail buffering policy imposes a fair Nash equilibrium. However, the Price of Anarchy is proportional to the number of users. It is also shown that the system converges to a Nash equilibrium in polynomial time, namely after $O(B^2)$ time steps. Then, the authors propose a simple modification of the Random Early Detection (RED) policy [FJ93] called Preemptive RED (PRED) that achieves a constant Price of Anarchy. It is also demonstrated that a Nash Equilibrium can be reached if all users deploy TCP Vegas as their transport protocol.

Finally, some natural extensions of the model including the case of multiple QoS requirements, routing on parallel links and general networks with multiple bottlenecks, are considered in [KSLB05]. It is shown that if users have different QoS requirements, the buffer usage of each user in a Nash equilibrium depends on the requested QoS. In the case $m$ identical links, the Price of Anarchy is shown to drop to $n/m$. It is also established that a max-min fair rate allocation is a Nash Equilibrium for general networks (under some restricting assumptions). The theoretical analysis of the equilibria is also complemented with simulations.

### 5 Algorithmic Mechanism Design

Mechanism Design is a subfield of Game Theory and Microeconomics which deals with the design of protocols for rational agents. Generally, a Mechanism Design problem can be described as the task of selecting from a collection of feasible games, a game which will yield desirable results for the designer. Specifically, the theory of Mechanism Design has focused on problems where the goal is to satisfactorily aggregate privately known preferences of several agents towards a social choice. Intuitively, a Mechanism Design problem has two components: the usual algorithmic output specification, and descriptions of what the participating agents want, formally given as utility functions over the set of possible outputs (outcomes).

A mechanism solves a given problem by assuring that the required outcome occurs, when agents choose their strategies as to maximize their own selfish utilities. A mechanism needs thus to ensure that players’ utilities (which it can influence by handing out payments) are compatible with the algorithm.

As mentioned earlier in this deliverable, the routing problem in large-scale networks, where users are instinctively selfish, can be modeled as a non-cooperative game. Such a game could impose strategies that might induce an equilibrium closer to the overall optimum. These strategies are formulated through pricing mechanisms [FCHPSS00], algorithmic mechanisms [NR99] and network design [KLO95, Rou01a]. The network administrator (or designer) can define prices or rules, or even construct the network in a way that induces near optimal performance when the users act selfishly.

In the network design approach, the network manager takes part in the noncooperative game. The manager has the ability to control centrally a part of the system resources, while the rest of the resources are used by the selfish users. This approach has been studied through Stackelberg or Leader-Follower games [BO99, KLO97, CDR03]. A selfish routing game can be also modeled as a Stackelberg game [Rou01b]. We here overview some issues related to how should the manager assign the flow he controls into the system, with the
objective to induce optimal cost due to the behavior of the selfish users.

5.1 Stackelberg Games

In [Rou01b], Roughgarden studies the problem of optimizing the performance of a system shared by selfish, noncooperative users assigned to shared machines with load-dependent latency functions. Roughgarden measures system performance by the total latency of the system. Assigning jobs according to the selfish interests of individual users typically results in suboptimal system performance. However, in many systems of this type there is a mixture of selfishly controlled and centrally controlled jobs; as the assignment of centrally controlled jobs will influence the subsequent actions by selfish users, the degradation in system performance due to selfish behavior can be reduced by scheduling the centrally controlled jobs in the best possible way.

[Rou01b] formulates this goal as an optimization problem via Stackelberg games, games in which one player acts a leader (here, the centralized authority interested in optimizing system performance) and the rest as followers (here, the selfish users). The problem is then to compute a strategy for the leader (a Stackelberg strategy) that induces the followers to react in a way that (at least approximately) minimizes the total latency in the system. Roughgarden [Rou01b] proves that it is NP-hard to compute the optimal Stackelberg strategy and present simple strategies with provable performance guarantees. More precisely, Roughgarden gives a simple algorithm that computes a strategy inducing a job assignment with total latency no more than a constant times that of the optimal assignment of all of the jobs; in the absence of centrally controlled jobs and a Stackelberg strategy, no result of this type is possible. Roughgarden also proves stronger performance guarantees in the special case where every machine latency function is linear in the machine load.

5.1.1 Stackelberg Equilibria

In [KKPS05], a selfish routing games is viewed as a Stackelberg game. The authors investigate issues related to how the manager assign the flow he controls into the system so as to induce optimal cost by the selfish users. In particular, they consider random tuples of machines, with either linear or M/M/1 latency functions, and Price of Anarchy at least a tuning parameter $c$ when there is no interference of the network designer (whom they call the Leader).

A variant (NLS) of the Largest Latency First (LLF) Leader’s strategy on tuples with Price of Anarchy $\geq c$ is evaluated. It is discovered that NLS experimentally improves on LLF for systems with inherently high Price of Anarchy, where the Leader is constrained to control a low portion $a$ of jobs. This suggests even better performance for systems with arbitrary Price of Anarchy. Also, the least Leader’s portion $a_0$ needed to induce optimum cost is bounded experimentally. Unexpectedly, as the parameter $c$ increases the corresponding $a_0$ decreases, for M/M/1 latency functions. All these are implemented in an extensive Matlab toolbox.

Assume now that we have a system $M$ of parallel machines with load dependent latency functions shared by an infinite number of users, each scheduling its infinitesimal small portion of total flow $r$ to machines in $M$ of currently minimum delay. This may yield a Nash assignment of flow with cost arbitrarily larger than the optimum one, increasing the Price of Anarchy on $M$. A Leader can decrease the Coordination Ratio on the price of the flow he controls. He wisely acts first, assigning flow $ar$ on $M$, and then all Followers react selfishly, assigning $(1-a)r$ flow on $M$. This is a Stackelberg Scheduling Instance $(M, r, a), 0 \leq a \leq 1$. The NP-hard problem $\begin{array}{c} 1 \\ 3 - 2 \\ 3 \end{array}$ PARTITION was reduced to deciding if
an instance $(M, r, a)$, restricted to linear latencies, admits a Leader’s strategy inducing a given cost. Thus, computing the optimal Leader’s strategy is NP-hard.

For systems with arbitrary latencies the Coordination Ratio has been shown to be at most $\frac{1}{4}$, or at most $\frac{1}{3\pi a}$ for linear latencies, or even better. For any system $M$ with arbitrary strictly increasing latencies, in [KKPS05] the authors efficiently compute the minimum portion $\beta_M$ of flow $r$ needed for a Leader to induce $M$'s optimum cost, and the Leader’s optimal strategy as well. Then, it is proved that computing the optimal Leader’s strategy on instances $(M, r, A \geq \beta_M)$ can be solved in polynomial time, and the Coordination Ratio is then 1. The really hard instances are $(M, r, a < \beta_M)$, where no Leader’s strategy can induce the optimum cost (so the Coordination Ratio is greater than 1).

Finally, the same work investigated instances with special latencies, where, the optimal Leader’s strategy can be computed in polynomial time. (This had been shown in the literature for $M/M/1$ latencies.) It is shown there that the same holds if each $M_i \in M$ incurs latency $\ell_i(x) = a_i x + b_i$, with $b_1 \leq \cdots \leq b_m$ and $a_1 = \cdots = a_m$.

5.1.2 The Price of Optimum

In [KPS05], a system $M$ of parallel machines, each with a strictly increasing and differentiable load dependent latency function is considered. The users of such a system are of infinite number and act selfishly, routing their infinitesimally small portion of the total flow $r$ they control, to machines of currently minimum delay. In that work such a system is modeled as a Stackelberg or Leader-Followers game motivated by [RT02].

In [Rou01b], Roughgarden presented the LLF Stackelberg strategy for a Leader, on a noncooperative game with an infinite number of Followers, each routing its infinitesimal flow through machines of currently minimum delay (this is the Flow Model from [Rou01b]). An important question posed there was the computation of the least portion $\beta_M$ that a Leader must control in order to enforce the overall Optimum Cost on a system $M$. In [KPS05], an algorithm that computes $\beta_M$ was presented and its optimality was also shown. Most importantly, it was proved that the algorithm presented is optimal for any system $M$ with any class of latency functions for which Nash and optimum assignments can be efficiently computed.

5.2 Cost Sharing Mechanisms

A simple and intuitive cost mechanism which assigns costs for the competitive usage of $m$ resources by $n$ selfish agents was proposed in [MPS05a]. Each agent has an individual demand; demands are drawn according to some probability distribution. The cost paid by an agent for a resource he chooses is the total demand put on the resource divided by the number of agents who chose that same resource. So, resources charge costs in an equitable, fair way, while each resource makes no profit out of the agents. This simple model was called Fair Pricing model by the authors of [MPS05a]. Its fair cost mechanism induces a non-cooperative strategic game, which is called FairPricingGame, whose players and strategies are the agents and resources, respectively.

In [MPS05a], the authors analyzed the Nash equilibria (both pure and mixed) for FairPricingGame; roughly speaking, these are stable states from which no agent has an incentive to unilaterally deviate. In particular, they consider the fully mixed Nash equilibrium where each agent selects each resource with non-zero probability. While offering in addition an advantage with respect to convenience in handling, the fully mixed Nash equilibrium is suitable for that economic framework under the very natural assumption that each resource offers usage to all agents without imposing any access restrictions.
To evaluate the Nash equilibria of this game, they introduced the Diffuse Price of Anarchy, as an extension of the Price of Anarchy that takes into account the probability distribution on the demands. Roughly speaking, the Diffuse Price of Anarchy is the worst-case, over all allowed probability distributions, of the expectation (according to each specific probability distribution) of the ratio of Social Cost over Optimum in the worst-case Nash equilibrium.

The Fair Pricing model, introduced by [MPS05a], was originally motivated by the standard KP-model for selfish routing; it departs from it by encompassing some stochastic assumptions on user demands, and notions of pricing and fairness as well.

In the same work, it was proved that pure Nash equilibria may not exist, unless all chosen demands are identical; in contrast, a fully mixed Nash equilibrium exists for all possible choices of the demands. Further on, it was proved that the fully mixed Nash equilibrium is the unique Nash equilibrium in case there are only two agents. It was also shown that, in the worst-case choice of demands, the Price of Anarchy is $\Theta(n)$; for the special case of two agents, the Price of Anarchy is less than $2 - \frac{1}{n}$. Assume now that demands are drawn from a bounded, independent probability distribution, where all demands are identically distributed and each is at most a (universal for the class) constant times its expectation. Then, it is proved in [MPS05a] that the Diffuse Price of Anarchy is at most that same constant, which is just 2 when each demand is distributed symmetrically around its expectation.

In a subsequent work [MPS05b], the authors consider two other cost sharing mechanisms which are closely related to the Fair Pricing model, namely the Average Cost Pricing and the Serial Cost Sharing models. There, it is proved that pure Nash equilibria do exist for both of these cost sharing mechanisms.

The Fair Pricing model in [MPS05a, MPS05b] provides a concrete first step toward a systematic way of treating such cost mechanisms for pricing the competitive usage of multiple resources in large-scale networks like the Internet.

6 Network Security Games

The recent huge growth of public networks (such as the Internet) has given to Network Security great importance and an even more critical role [Sta03]. The highly dynamic, distributed nature and the huge size of public networks and future networks impose the need of new advances, methods and tools for the efficient management of network security issues. Most current and future networks may not be considered to be absolutely secured. The system security software can no longer be assumed to have a global access to the network; rather, at any time, insecure network entities may join the network causing a possible viruses infection.

[ACY05] and [KO04] studied network security problems modeling them as Interdependent Security games. In such a game, a large number of players must make individual investment decisions related to security, in which the ultimate safety of each participant may depend in a complex way on the actions of the entire population. In [ACY05], the authors establish connections of the game considered with variants of a Graph Partition problem. Using them they provide polynomial-time computable Nash equilibria and prove $NP$-hardness of finding the best equilibria (with respect to a suitable Social Cost they defined).

The work of [FGY00] studies the feasibility and computational complexity of two privacy tasks in distributed environments with mobile eavesdroppers: those of distributed database maintenance and message transmission. A mobile eavesdropper is a computationally unbounded adversary that move its bugging equipment within the system. However, this
work does not utilize Graph-Theoretic tools. In contrast, [AKPW95] employs Graph-Theoretic tools to study a two-player game on a graph. It establishes connections of the problem with the $k$-server problem and provides an approximate solution for a simple associated network design problem. However, this study does not concern network security problems.

[MPPS05b] considers a security problem on a distributed network modeling it as a multi-player non-cooperative game with attackers (e.g., viruses) and a defender (e.g., a security software) entities. The authors exploit both game-theoretic and graph-theoretic tools for studying the associated Nash equilibria.

### 6.1 A Graph-Theoretic Model

[MPPS05b], considers a distributed network whose nodes are insecure and vulnerable to infection by harmful entities (e.g., viruses, worms, trojan horses, eavesdroppers [FGY00]), called the attackers. A system security software, the defender, is available in the system. However, due to the network size and for economic and performance reasons, it is capable to provide safety (that is, clean nodes from the possible presence of attackers) only in a limited part of the network (e.g., a small set of edges).

At any time, attackers and the defender take individual decisions, based on limited information due to the distributed nature of the system, for their placement in the network, seeking to maximize their (opposite) objectives. In particular, each attacker targets a location (i.e., a node) of the network via a probability distribution; the node is damaged unless it is cleaned by the defender. The defender is able to clean a link or a set of links, which it may choose using a probability distribution. Such limitations are due to financial costs, such as the cost of purchasing a global security software, or due to performance reasons, such as the reduced efficiency or usability of the protected network part. The selection of the defender seeks to protect the network as much as possible, while the harmful entities wish to avoid being caught so as to be able to damage the network. Thus, it is reasonable to view the problem as a non-cooperative game with players of conflicting interests.

[MPPS05b] assumes the most basic case of the problem scenario, where the defender is able to clean only a single edge of the network. Then, the problem is modeled as a non-cooperative, multi-player strategic game played on a graph. The game is played on a graph with two kinds of players: the vertex players, which can choose a node of the network, represent the attackers, and the defender, called the edge player, which can choose a single edge of the network and represent the system security software. The defender seeks to maximize the expected number of vertex players it catches, while each vertex player seeks to maximize the probability of escaping the defender. The resulting game is called the Edge model.

Such a modeling captures the simplest case of the problem. At the same time, its simplicity enables a relative ease for exploring the problem using graph-theoretic tools. For this game, the associated Nash equilibria, where no network entity can unilaterally improve its local objective are of interest.

For the edge model in [MPPS05b], it is proved that no instance of the game has a pure Nash equilibrium. The authors then proceed to study mixed Nash equilibria. A graph-theoretic characterization of mixed Nash equilibria is provided. Roughly speaking, the characterization implies that the support of the edge player and the vertex players are an edge cover and a vertex cover of the graph. Given the supports, the characterization provides a system of equalities and inequalities to be satisfied by the probabilities of the players. Unfortunately, this characterization only implies an exponential time algorithm
for the general case.

The same work introduced a subclass of Nash equilibria, called matching Nash equilibria, which are a natural subclass of mixed Nash equilibria with a graph-theoretic definition. Roughly speaking, the supports of vertex players in a matching Nash equilibrium form together an independent set of the graph, while each vertex in the union of supports of the vertex players is incident to only one edge from the support of the edge player.

A characterization of graphs admitting a matching Nash equilibrium is then provided. It is proved that a matching Nash equilibrium can be computed in linear time for any graph satisfying the characterization once a suitable independent set is given for the graph. The authors proceed to consider bipartite graphs for which they show that they satisfy the characterization of matching Nash equilibria; hence, they always have one. More importantly, it is proved that a matching Nash equilibrium can be computed in polynomial time for bipartite graphs.

In a subsequent work [MPPS05a], the authors proceed to studying the same problem on other graph families. Utilizing graph-theoretic arguments and the characterization of mixed Nash equilibria proved in their earlier work, they show how to compute in polynomial time mixed Nash equilibria on corresponding graph instances. The graph families considered are regular graphs, graphs with polynomial time computable $r$-regular factors, and graphs with perfect matchings.

In the same work, the Social Cost of the game is defined to be the expected number of attackers caught by the protector. It is proved that the corresponding Price of Anarchy in any mixed Nash equilibria of the Edge model is upper and lower bounded by a linear function of the number of vertices of the graph. Finally, [MPPS05a] considers a more generalized variation of the problem, captured by the Path model. It is there proved that the problem of existence of a pure Nash equilibrium is $NP$-complete for this model.

6.2 A Generalized Model

[GMP05+] introduced and studied a generalized version of the network security problem considered in [MPPS05b]. In more detail, the authors in [GMP05+] consider a more general case of the Edge model where the defender is able to scan a set of $k$ links of the network; this generalization is called the Tuple model. It is natural to expect that this increased power of the defender should result in a better quality of protection for the network. Ideally, this would be achieved at little expense on the existence and complexity of Nash equilibria.

In that work, it is proved that the existence problem for pure Nash equilibria is solvable in polynomial time. Then, the authors provide a graph-theoretic characterization of mixed Nash equilibria.

Inspired by matching Nash equilibria, the authors of [MPPS05b] introduce $k$-matching configurations that generalize matching configurations. They provide a polynomial-time reduction for transforming any matching Nash equilibrium of any instance of the Edge model to a $k$-matching Nash equilibrium on a corresponding instance of the Tuple model, and vice versa. Using the polynomial-time reduction, they provide a characterization of graphs admitting $k$-matching Nash equilibria. Furthermore, a polynomial-time algorithm for computing $k$-matching Nash equilibria on graphs satisfying the characterization is provided. The applicability of the algorithm is demonstrated for the case of bipartite graphs.

In the same work, it is established that the increased power of the defender results in an improved quality of protection of the network. In particular, for the case of $k$-matching Nash equilibria, it is proved that the gain of the defender, which amounts to the expected
number of the arrested harmful procedures, increases linearly with the parameter \( k \) (the number of network links the defender is able to scan and protect).

7 The Core

In the recent years, there has been an increased interest in computational complexity aspects of solution concepts in Cooperative Game Theory, such as the \textit{nucleolus} of a game. On the positive side, efficient algorithms have been developed for the computation of the nucleolus of assignment games [SR94], the nucleus of matching games [FFHK98], and the nucleolus of min-cost spanning tree games [Meg87]. On the negative side, several NP-hardness results were obtained; for example for testing core membership [FFHK97] or computing the nucleolus for min cost spanning tree games [FKK98] have been proven to be NP-hard. It is a challenging problem in Mathematical Programming to characterize the classes of cooperative games that permit polynomial time computation of the nucleolus.

A notion related to the nucleolus is that of the \textit{core}. The core is a solution that possesses a sorted excess vector with all components non-negative. It can be represented by one single linear program (of exponential size). For the computation of the core, the linear programming approach has been very successful. A central approach is to establish an equivalent \textit{polynomial-size} integer program with a polynomial time solution; see, for example, the linear production game of Owen [Owe75], the partition game of Faigle and Kern [FK95], the packing/covering game of Deng, Ibaraki and Nagamochi [DIN97] and the facility location game of Goemans and Skutella [GS00]. Linear program duality has played an important role in the above results characterizing the core in Cooperative Game Theory.

A very interesting \textit{flow game} was introduced by Kalai and Zemel [KZ82b, KZ82a], which arose from the profit distribution problem related to the maximum flow in a network where arcs are owned by different individuals. It was shown in [KZ82b, KZ82a] that the flow game on a simple network (with edge capacities all equal) is totally balanced, and the allocations corresponding to minimum cuts in the network always belong to the core. (Linear program duality was crucial for proving the results.) They further conjectured that their approach can lead to an efficient algorithm for the computation of the nucleolus.

In [DFS06], the nucleolus of flow games is studied from the algorithmic point of view. There it is shown that computing the nucleolus can be done in polynomial time for the flow game on a simple network. The proof is deep and an elegant application of linear program duality approach in Kalai and Zemel’s work [KZ82a], thus settling their conjecture. On the other hand, the authors prove that both the computation and the recognition of the nucleolus are NP-hard for flow games in general cases. The NP-hardness proof for recognizing the nucleolus also resolves a conjecture of Faigle, Kern and Kuipers [FKK98].

8 Complexity of Computing Equilibria

8.1 Pure Nash Equilibria in Multi-players Games

A core question in the study of Nash equilibria is which games have pure Nash equilibria. Also, under what circumstances can we find one in polynomial time? Note that for such a problem to be computationally meaningful, the number of players should be large and the payoff table must be given in some implicit way (e.g., in a succinct representation). As mentioned earlier, congestion games is a class of games that are guaranteed to have pure Nash equilibria.
In a classical paper [Ros73], Rosenthal proves that, in any such game, the Nash dynamics converges, equivalently, the directed graph with action combinations as nodes and payoff improving defections by individual players as edges is acyclic. Hence, the game has pure Nash equilibria which are the sinks of the graph. The proof is based on a simple potential function. This existence theorem, however, again leaves open the question of whether a polynomial-time algorithm for finding pure Nash equilibria in congestion games exists.

In [FCPT04], it is shown that the answer is positive when all players have the same origin and destination (the so-called, symmetric case); a pure Nash equilibrium is found by computing the optimum of Rosenthal's potential function through a reduction to min-cost flow. However, it is shown that computing a pure Nash equilibrium in the general network case is \textit{PLS}-complete [JPY88]. Intuitively, this means that it is as hard to compute as any object whose existence is guaranteed by a potential function. The proof of [FCPT04] has as corollary the existence of examples with exponentially long shortest paths, as well as the \textit{PSPACE}-completeness of the problem of finding a Nash equilibrium reachable from a specified state.

The completeness proof requires reworking the reduction to the problem of finding local optima of weighted MAX2SAT instances. When congestion games are posed in the abstract (in terms of sets of resources instead of paths in a network, this being the original formulation), Nash equilibria are \textit{PLS}-complete to find even in the symmetric case.

The algorithm presented there for pure Nash equilibria has an application to the non-atomic congestion games studied by Roughgarden and Tardos [RT], in which delays are continuous functions. In [FCPT04] it is also shown that under some necessary smoothness assumptions, we can approximate the Nash equilibria of such games in strongly polynomial time.

### 8.2 Games with at Least Four Players

Daskalakis et al. [DGCHP05] consider the question of the complexity of Nash equilibrium in a game with four or more players. They show that this problem is complete for the complexity class \textit{PPAD}. Thus, a polynomial-time algorithm would imply a similar algorithm, e.g., for computing Brouwer fixpoints, a problem for which quite strong lower bounds for large classes of algorithms are known [HPV89], and would have to fail to relativize with respect to the oracles in [BCE+98], for which \textit{PPAD} has no polynomial-time algorithms. As it is well shown, Nash showed his celebrated result by reducing the existence of Nash equilibria to the existence of Brouwer fixpoints. Given any strategic game, he constructs a Brouwer function whose fixpoints are precisely the equilibria of the game. In Nash’s reduction, as well as in subsequent ones [Gea03], the constructed Brouwer function is quite specialized, and this has led to speculation on whether the fixpoints of such functions (and thus, Nash equilibria) are easier to find than for general Brouwer functions. In [DGCHP05], this question is answered in the negative by presenting a reduction in the opposite direction: any (computationally presented) Brouwer function can be simulated by a suitable game, so that Nash equilibria correspond to fixpoints.

### 8.3 Games with Three Players

In [DGCHP05], it is proved that computing a Nash equilibrium in a 3-player game is also \textit{PPAD}-complete. The proof is based on a variant of the arithmetical gadget of [GP05], containing few extra nodes. The new gadget, when used in the reduction from Brouwer’s problem in [DGCHP05], has the effect of creating a graphical game whose moralized graph is 3-colorable, thus reducing the number of players needed to simulate the graphical game to three.
Independent, Chen and Deng [CD05a] have also come up with a proof of the same result. As noted in [DCHP05], the proof is quite different from that of [DCHP05]: Without changing the gadgets, they come up with a more sophisticated version of the simulation in [GP05] in which just three players are needed to simulate a general game (despite the fact that four colors are required to color the moralized graph). They do this by exploiting a subtle form of disconnectedness in the graphical game constructed in [GP05].

8.4 Games with Two players

An important, long-standing open problem in the boundary of Game Theory and Theoretical Computer Science has concerned the complexity of finding a Nash equilibrium in a game with two players. As discussed above, finding a Nash equilibrium in a game with three players is \( PPAD \)-complete. Finding a Nash equilibrium in a game with two players could be easier for several reasons. For example, an important technique employed in the hardness proofs, that of coloring vertices of a graphical game, does not seem possible to work to the case of two players.

Very recently, Chen and Deng [CD05b] settle the problem with a \( PPAD \)-completeness proof for the 2-player Nash equilibrium problem. Their proof got rid of the graphical game model and derived a direct reduction from a search problem called the 3-DIMENSIONAL BROUWER problem, which is known to be \( PPAD \)-complete [DGCHP05] to the objective problem. The completeness proof of [CD05b] utilizes new breakthrough gadgets for various arithmetic and logic operations which are successfully designed. The paper of Chen and Deng settles the complexity of games with two players.

8.5 Correlated Equilibria

The Nash equilibrium [Nas50, Nas51] is considered to be the standard notion of rationality in Game Theory. However, there are several other competing notions of rationality; chief among them is the correlated equilibrium, proposed by Aumann [Aum74]. If the mixed Nash equilibrium is a distribution on the strategy space that is uncorrelated or independent (that is, the product of independent distributions, one of each player), a correlated equilibrium is a general distribution over strategy profiles. It must, however, possess an equilibrium property: If a strategy profile is drawn from this distribution (presumably by a trusted external agent), and each player is told separately his/her own component, then no player has an incentive to choose a different strategy because, assuming that all other players also obey, the suggested strategy is the best in expectation.

As noted by Papadimitriou in [Pap05], the correlated equilibrium has several important advantages: It is a perfectly reasonable, simple and plausible concept; it is guaranteed to always exist (simply because the Nash equilibrium is a particular case of a correlated equilibrium); and it can be found in polynomial time for any number of players and strategies by linear programming, since the inequalities specifying the quiescence property above are linear. In fact, the correlated equilibrium that optimizes any linear function of the players’ utilities (for example, their sum) can be computed this way.

Succinct Games  Equilibrium problems in games, of which the correlated equilibrium is a quite prominent example, are objects worth of studying from the algorithmic point of view. Multiplayer games are the most compelling specimens in this regard. But, to be of interest, they must be somehow represented succinctly. Succinct representation is required since otherwise a typical (multiplayer) game, would need exponential size of bits in order to be described. Some well known games that admit a succinct representation include:
• symmetric games, where all players are identical and indistinguishable,
• graphical games, for which the players are the vertices of a graph and the payoff of each player only depends on its strategy and those of its neighbours;
• congestion games, where the payoff of each player only depends on its strategy and those playing on the same strategy as him.

Papadimitrou and Roughgarden [PR05] initiated the systematic study of algorithmic issues involved in finding equilibria (both Nash and correlated) in games with a large number of players which are succinctly specified. The authors develop a general framework for obtaining polynomial-time algorithms for optimizing over correlated equilibria in such settings. They show how it can be applied successfully to symmetric games (for which we actually find an exact polytopal characterization), graphical games, and congestion games, among others. They also present complexity results, implying that such algorithms are not possible for certain other similar games. Finally, a polynomial-time algorithm, based on quantifier elimination, for finding a Nash equilibrium in symmetric games when the number of strategies is relatively small was presented.

Daskalakis and Papadimitriou [DCP05] studied from the complexity point of view the problem of finding equilibria in games defined by highly regular graphs with extremely succinct representation, such as the $d$-dimensional grid. There, it is argued that such games are of interest in modeling large systems of interacting agents. It was shown that the problem of determining whether such a game on the $d$-dimensional grid has a pure Nash equilibrium depends on $d$, and the dichotomy is remarkably sharp: It is polynomially time solvable when $d = 1$, but $NEXP$-complete for $d \geq 2$. In contrast, it was proved that mixed Nash equilibria can be found in deterministic exponential time for any $d$ by quantifier elimination.

Recently, Papadimitrou [Pap05] considered, and largely settled, the question of the existence of polynomial-time algorithms for computing correlated equilibria in succinctly representable multiplayer games. The author developed a polynomial-time algorithm for finding correlated equilibria in a broad class of succinctly representable multiplayer games, encompassing essentially all known kinds.

The algorithm presented was based on a variant of the existence proof due to Hart and Schmeidler [HS89], and employs linear programming duality and the ellipsoid algorithm, Markov chain steady state computations, as well as application-specific methods for computing multivariate expectations.

9 Discussion

In this document, we overviewed some of the most important advances related to approximating net equilibria, net flows and the Core, Price of Anarchy, for restricted and unrestricted agent rationality. However, a number of open problems on most of the research areas addressed remain still open. Such issues cover almost all areas considered and include:

• The further exploration of the structure, complexity, and efficient computation of Nash equilibria for routing problems in large-scale, complex and dynamic networks. The same research goals constitute also a great challenge for some other practical, important network tasks, such as security and communication.
• The further estimation (via upper and lower bounds) of the Price of Anarchy for selfish routing games and for other fundamental tasks, such as security and communication, is still a wide-open research field.
• The structure and complexity of Nash equilibria in selfish routing games with incomplete information is calling for a thorough investigation.

• The further development of Algorithmic Mechanism Design remains a great challenge for the designers of emerging complex networks such as the Internet.

References


