The chromatic and clique numbers of random scaled sector graphs

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2004
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Abstract

Random scaled sector graphs were introduced as a generalization of random geometric graphs to model networks of sensors using optical communication. In the random scaled sector graph model vertices are placed uniformly at random into the $[0,1]^2$ unit square. Each vertex $i$ is assigned an uniformly at random sector $S_i$, of central angle $\alpha_i$, in a circle of radius $r_i$ (with vertex $i$ as the origin). An arc is present from vertex $i$ to any vertex $j$, if $j$ falls in $S_i$. In this work, we study the value of the chromatic number $\chi(G_n)$, directed clique number $\omega(G_n)$, and undirected clique number $\hat{\omega}_2(G_n)$ for random scaled sector graphs with $n$ vertices, where each vertex spans a sector of $\alpha$ degrees with radius $r_n = \sqrt{\frac{\ln n}{n}}$. We prove that for values $\alpha < \pi$, as $n \to \infty$ w.h.p., $\chi(G_n)$ and $\hat{\omega}_2(G_n)$ are $\Theta(\ln \ln n)$, while $\omega(G_n)$ is $O(1)$, showing a clear difference with the random geometric graph model. For $\alpha > \pi$ w.h.p., $\chi(G_n)$ and $\hat{\omega}_2(G_n)$ are $\Theta(\ln n)$, being the same for random scaled sector and random geometric graphs, while $\omega(G_n)$ is $\Theta(\ln \ln n)$.

1 Introduction

Massive networks of wireless sensors are known to play an important role in monitoring and disseminating information [ASSC02, AB02]. The general setting of such a network is to have a large collection of wireless motes (sensors) randomly scattered in a remote or hazardous terrain, performing tasks of distributed sensing. The sensing information gathered by the motes is relied to a base station. To communicate, either among themselves or with a monitoring base station, the motes use radio-frequency (RF) or optical communication. In the RF communication model, the motes either use an omnidirectional antenna, which spreads the signal in a spherical region centered at the antenna, or a directional antenna, which has a focused beam spanning a sector of $\alpha$ degrees. In sensor networks, directional antennas have multiple advantages over omnidirectional antennas: less energy consumption, less fading area, and furthermore as the transmission area is smaller the channel interference may have less influence [BGL02]. In the optical communication model motes can send information using...
an orientable laser beam embedded with an optical receiver. In this model motes can receive information from any mote within a prescribed distance whose transmitting laser is orientated towards them [KKRP99].

In recent times, there has been an effort to provide a theoretical framework to study networks of sensors. For the omnidirectional RF communication network, a suitable model is the random geometric graph, also denoted random scaled disk graph. These graphs are the random scaled version of the unit disk graphs described in [CCJ90]. The model considers the network as a graph scaled into [0, 1]^2, where the n random deployed motes are the vertices of a random graph in [0, 1]^2, and two vertices are connected if they are a within Euclidean distance r_n, corresponding to the broadcast range of the motes. Many results are known about the properties of random scaled disk graphs. For instance, when r_n = \sqrt{\ln n / n}, it is known the chromatic number \chi and the clique number \omega are asymptotically \Theta(ln n) (see [Pen03]).

The natural model for the case of directional RF and optical networks seems to be the random scaled sector graph, a generalization of the random geometric graph, introduced in [DPS03]. In the setting under consideration, the motes have a fixed angle \alpha (0 < \alpha \leq 2\pi) of maximum scanning, defining a sector of transmission, moreover when mote i falls in [0, 1]^2, there will be an angle between the beam and the horizontal axis. We represent this angle as a random variable \beta_i giving the “elevation” of i with respect to the horizontal direction. We represent the beam emitted by i as the sector S_i, centered at i, with radius r, amplitude \alpha and elevation \beta_i. Every other sensor which falls inside of S_i can potentially receive the signal emitted by i (see Fig. 1). The random scaled sector graph is the graph with vertices as the sensors, in which there is an arc from i to j if j falls inside S_i, (see formal definition in Section 2). Some of the graph parameters for sector graphs coincide with the ones for geometric graphs, for instance in both graphs the threshold for connectivity, in terms of the distance r is r_n = \Theta(\sqrt{\ln n / n}) (see [DPS03]). It should be noted, that in practical applications, the values of \alpha are small, typically from \pi/20 to \pi/4, depending on the type of communication (RF or optical).

In this paper we study the value of the chromatic number \chi(G_n), directed clique number \omega(G_n), and undirected clique number \hat{\omega}_2(G_n) for random scaled sector graphs with n vertices and radius r_n = \sqrt{\ln n / n}. We prove that for values \alpha < \pi, as n \to \infty w.h.p., \chi(G_n) is \Theta(\ln n), showing a clear difference with the random geometric graph model, which is \Theta(ln n), as we already mentioned. For \alpha > \pi, w.h.p. the value of \chi(G_n) is \Theta(ln n) for both random sector and geometric graphs.

2 Results

A random scaled sector graph is defined in the following way,

**Definition 1 ([DPS03])** Assume that the angle \alpha is a fixed parameter of the sensors. Let X = (x_i)_{i \geq 1} be a sequence of independently and uniformly distributed (i.u.d.) random points in [0, 1]^2, let B = (\beta_i)_{i \geq 1} be a sequence of i.u.d. angles and let R = (r_i)_{i \geq 1} be a sequence of numbers in [0, 1]. We write X_n = \{x_1, \ldots, x_n\} and B_n = \{\beta_1, \ldots, \beta_n\}. We call the digraphs G_n = G_{\alpha}(X_n, B_n, r_n) the random scaled sector graph on n nodes, where V(G_n) = X_n and the arcs are defined by: (x_i, x_j) \in E(G_n) iff x_j \in S_i.
We use the letter $H$ to denote a subgraph of $G_n$. $\Delta$, denotes the maximum degree of $G_n$. Given $G_n$, as usual the chromatic number, and the size of the maximum directed clique, are represented by $\chi(G_n)$ and $\omega(G_n)$, respectively. Since we are dealing with directed graphs, we introduce a new variable $\hat{\omega}_2$, which represents the size of the maximum undirected clique, where for any two vertices $u, v \in V(G_n)$, to be members of the same undirected clique, only one of the two possible arcs, $(u, v)$ or $(v, u)$, need be present in the graph $G_n$. Thus, $\omega(G_n) \leq \hat{\omega}_2(G_n) \leq \chi(G_n)$, and for $\alpha = 2\pi$, $\omega(G_n) = \hat{\omega}_2(G_n)$.

We say $G_n$ has a property $T$, with high probability (w.h.p), if as $n \to \infty$, we expect $G_n$ to have property $T$, with probability $1 - O(1/n^c)$, for some $c > 0$. For other concepts and results in probability theory, look for example [Pen03].

In the remainder of the paper we prove the following results for $r_n = \sqrt{\frac{\ln n}{n}}$.

**Theorem 1** Let $\epsilon > 0$. For $\epsilon < \alpha < \pi - \epsilon$, the size of the maximum directed clique, $\omega(G_n)$ is $\Theta(1)$. For $\pi + \epsilon < \alpha < 2\pi - \epsilon$, w.h.p., $\omega(G_n)$ is $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$.

**Theorem 2** Let $\epsilon > 0$. For $\epsilon < \alpha < \pi - \epsilon$, w.h.p., the chromatic number, $\chi(G_n)$ is $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$. For $\pi + \epsilon < \alpha$, w.h.p., $\chi(G_n)$ is $\Theta(\ln n)$.

**Theorem 3** Let $\epsilon > 0$. For $\epsilon < \alpha < \pi - \epsilon$, w.h.p., the size of the maximum undirected clique, $\hat{\omega}_2(G_n)$ is $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$. For $\pi + \epsilon < \alpha$, w.h.p., $\hat{\omega}_2(G_n)$ is $\Theta(\ln n)$.

## 3 Basic constructions and lemmas

In this section, we present some tools and lemmas, which are needed to prove Theorems 1, 2, and 3. In order to lighten the notation we define the following variables,

$$ a_n = \frac{\ln n}{\ln \ln n} \quad \text{and} \quad b_n = \sqrt{n \ln n}. $$

Recall, the orientation angle, $\beta_i$, of every mote $i$ is drawn uniformly at random (u.a.r) from $(0, 2\pi]$. Many of the proofs in this paper require partitioning the orientation angle into classes. Thus we define a partition $\mathcal{B}$ of the orientation angle as follows.

![Figure 1: The sector of a sensor $i$ and the communication between motes](image)
Figure 2: Angle partition for $\alpha > \pi + \epsilon$ (a) classes $B$ (b) directions associated to a class $B_j$

Figure 3: The basic dissections of $[0, 1]^2$ (a) $S$ (b) horizontal subdivision (c) vertical subdivision

**Definition 2** Let $\epsilon$ be a constant (depending on $\alpha$), such that $\alpha = \pi + \epsilon$, for $0 < \epsilon < \pi$. A $B$ partition, is a partition of the region $2\pi$ into $B$ classes, each of length $\epsilon - 2\epsilon^*$, with $\epsilon^*$ a constant chosen such that $\epsilon > 2\epsilon^*$ (see Figure 2). All motes $i$ such that $\beta_i$ fall whose the same range will belong to the same class. More specifically, for any $1 < j \leq B$, the class $B_j$ is defined as the class of motes whose bisectrix falls between $(-\frac{3}{2} + j)\epsilon - (2j - 3)\epsilon^*$ and $(-\frac{1}{2} + j)\epsilon - (2j - 1)\epsilon^*$. Notice $B = \lceil \frac{2\pi}{\epsilon - 2\epsilon^*} \rceil$, so $B \in \mathbb{Z}$.

Throughout the paper, when we refer to the dissection $S$ of $[0, 1]^2$, we mean a partition of $[0, 1]^2$ into $\frac{n}{\ln n}$ squares, each one of size $r_n \times r_n$. Also, in the paper we make use of two lemmas. The first one is proved via Chernoff bounds and Boole’s inequality,

**Lemma 1 ([DPS03])** If $n$ motes are distributed u.a.r. on $[0, 1]^2$, w.h.p. each of the squares in the dissection $S$ will contain $\Theta(\ln n)$ motes. In particular, w.h.p. every square in the dissection $S$ will contain at most $3 \ln n$ motes.

In order to prove the second lemma, we use an implication of Talagrand’s inequality, given in [MR00]:

**Talagrand’s Inequality** Let $X$ be a non-negative random variable, not identically 0, which is determined by $n$ independent trials $L_1, ..., L_m$, and satisfying the following for some $b, r > 0$:
1. Changing the outcome of any one trial can affect $X$ by at most $b$.

2. for any $s$, if $X ≥ s$ there is a set of $rs$ trials whose outcomes certify that $X ≥ s$.

Then, for any $0 ≤ l ≤ E[X]$, $P \left( |X - E[X]| > l + 60b√rE[X] \right) ≤ 4e^{-l^2/32rE[X]}$.

**Lemma 2** Given the dissection $S$ of $[0,1]^2$, divide each square of $S$ into $\ln n$ rectangular regions of size $\frac{n}{\ln n} × r_n$ (see Figure 3). Then, w.h.p. there exists at least one region $R_i$, which contains $(1 - o(1))\frac{a_n}{B}$ motes from every class $B_j ∈ B$.

**Proof** In order to prove Lemma 2 we make use of the following fact. If $n$ balls are dropped into $n$ bins, w.h.p. at least one bin contains $a_n = \ln n/\ln n$ balls. Notice by construction the number of regions in $[0,1]^2$ is $\frac{n}{\ln n} \ln n = n$, since the $n$ motes are distributed u.a.r. on $[0,1]^2$, by a balls-and-bins argument, there is a region $R_i$, which w.h.p. contains $a_n = \ln n/\ln n$ motes. Let $X_j$ be a random variable counting the number of motes in $R_i$, which are in class $B_j$. Then $E[X_j] = a_n/B$. To complete the proof of Lemma 2 we show via Talagrand’s inequality the random variable $X_j$ is concentrated around its expectation. First note, $X_j$ is determined by the $m = (1 - o(1))a_n/B$ trials specifying $\{\beta_1, \ldots, \beta_m\}$. Also changing the outcome of any one $\beta_l$, $1 ≤ l ≤ m$, can affect $X_j$ by at most one, and in order to certify $X_j ≥ s$, only the outcomes of $s$ trials (the $s$ $\beta_l$’s which fall in that class) are required. Thus the conditions of Talagrand’s inequality are satisfied with $b, r = 1$. Hence by Talagrand’s inequality and Boole’s inequality, w.h.p. every class contains $(1 - o(1))a_n/B$ motes. □

### 4 Proof of Theorem 1

#### 4.1 $α < π − ε$

**Proof** When $α < π − ε$ the vertices of any clique must form a convex polygon. This can be proved by first noting that in every clique of size three, the three points cannot be collinear, and proceeding inductively. Let $|V|$ represent the number of vertices in any clique. Since the sum of the angles of a convex polygon is $(|V| - 2)π$, we have $|V|α ≥ (|V| - 2)π$, thus $ω(G_n) ≤ \left\lfloor \frac{2π}{π - α} \right\rfloor$. □

#### 4.2 $α > π + ε$

**Proof** First we establish the lower bound, by proving a certain sufficient configuration of motes exists (w.h.p.). Consider the $S$ partition of $[0,1]^2$. Subdivide each small square into $\ln n$ equal (in terms of area) vertical regions (one can imagine drawing $\ln n$ equally spaced vertical lines). By Lemma 2, there is a vertical region $R_i$ w.h.p. containing $(1 - o(1))\frac{a_n}{B}$ motes who are members of the class $B_1$, i.e. the bisectrix of these motes is between $-\frac{1}{2}ε + ε^*$ and $\frac{1}{2}ε - ε^*$. Let $M_1$ be the set of these motes (in class $B_1$) in $R_i$. Further subdivide the region $R_i$ into $a_n/B$ cells, each cell a rectangle of width $\frac{1}{b_n}$ and height $\frac{Bc_n}{a_n}$, see Fig. 4. Let $Y$ be a random variable counting the number of cells containing at least one vertex from $M_1$, then $E[Y] = (1 - \frac{1}{r})a_n/B$ as $n → ∞$, and as in the proof of Lemma 2, one can show $Y$ is concentrated around its expectation by applying Talagrand’s inequality. Thus w.h.p there are at least $(1 - (1/4 + o(1)))a_n/B$ cells containing at least one mote from $M_1$. Consider a mote $m$ from the set $M_1$, due to its orientation angle, it will have an arc with any mote $m'$ which is
an cell more than a specific distance, \( l \), and within distance \( r_n \) in either direction, up or down from itself. Where \( l \) depends on the exact orientation angle of the mote \( m \), and the location of the two motes, \( m \) and \( m' \) in their respective cells. Consider the worst case, assume the mote \( m \) is in the lower right-hand corner of its cell and the mote \( m' \) is in the upper left-hand corner of its cell (see Fig. 4). In this case assuming \( m \) has a bisectrix of 0, by trigonometry, \\
\[ l = \frac{\cos((\alpha - \pi)/2)}{\sin((\alpha - \pi)/2)b_n}. \]
However \( m \) need not have a bisectrix of 0. Since \( m \) is in the class \( B_1 \), its bisectrix is between \(-\frac{1}{2} \epsilon + \epsilon^* \) and \( \frac{1}{2} \epsilon - \epsilon^* \), thus in the worst case, \\
\[ l = \frac{\cos((\alpha - \pi - \epsilon + 2\epsilon^*)/2)}{\sin((\alpha - \pi - \epsilon + 2\epsilon^*)/2)b_n}. \]
Recall \( \alpha > \pi + \epsilon \), thus when \( \alpha \) assumes its lowest value, \\
\[ l = \frac{\cos(\epsilon^*)}{\sin(\epsilon^*)b_n}. \] For small \( x \), \( \sin(x) \sim x \), given that \( \epsilon^* \) is a constant, \( l = c/b_n \), for some constant \( c \). Since the height of each cell is \( \frac{B_{\Omega \omega_n}}{a_n} \), w.h.p., \( \omega(G_n) \geq c'a_n \), for a sufficiently large constant \( c' \) dependent on \( \alpha \).

- Next we establish the upper bound, by showing w.h.p., a certain necessary configuration cannot exist. In order to prove the upper bound we make use of the following easily verified fact, let \( \omega^* \) represent the size of the largest directed clique in any square of \( S \), then the size of \( \omega \) is upper-bounded by \( \Theta(\omega^*) \). Thus to establish the upper bound, we will prove w.h.p., there exists a sufficiently large constant \( d \) such that no set of \( \frac{d}{B} a_n \) motes in any square \( S \) form a clique, i.e. \( \omega^* \leq \Theta(a_n) \) and the statement of the theorem will follow.

Again consider the partition \( \mathcal{S} \) of \([0, 1]^2\) and the \( B \)-partition. Fix any square \( S \in \mathcal{S} \). By Lemma 1, \( S \) contains at most \( 3 \ln n \) motes. Select u.a.r. \( d \cdot a_n \) motes from \( S \). By the Pigeon-hole principle, at least \( \frac{d}{B} a_n \) of those motes will have the bisectrix oriented into the same class, call this class \( B_j \in \mathcal{B} \). Let \( M_j \) be the set of all those motes in the class \( B_j \). Define a partition of \( S \) into \( \ln n \) strips in the following way: Imagine a mote exists in \( S \) whose bisectrix is orientated exactly in the center of the class \( B_j \). Draw a parallel line to the bisectrix of this (imaginary) mote. This (parallel) line will be the orientation of the strips. Cover \( S \) with \( \ln n \) rectangular strips parallel to the orientation (see Figure 5). For example, in the case where these motes belong to the class \( B_1 \), the rectangular strips will be parallel to the sides of \( S \).

Note by construction, in this partition of \( S \), the optical sensors of all the motes in the class \( M_j \) look in the same approximate direction. Thus for these motes to be a part of the same clique every mote must see all the other motes along some specified direction, and we will show w.h.p. this will not be the case.

Before we continue with the remainder of the proof we need to establish the following lemma.

**Lemma 3** For a sufficiently large but constant \( d \), any set of \( \frac{d}{B} a_n \) motes, will (w.h.p.) occupy at least \( (\ln n)^{9/10} \) strips.

**Proof of Lemma 3** First we upperbound the area of any strip. The maximum length any strip can have is \( r_n \sqrt{2} \) since we are considering a \([r_n \times r_n]\) square \( S \). The maximum width any strip can have is \( \sqrt{2} r_n / \ln n \) (this occurs when the orientation of strips is parallel to the diagonal of \( S \)). Thus the area of the largest strip is bounded above by \\
\[ \sqrt{2} r_n / \ln n \times r_n \sqrt{2} = \frac{2}{n}. \]

Now we upperbound the probability that any set of \((\ln n)^{9/10}\) of the strips will contain \( \frac{d}{B} a_n \) motes in \( S \).
Figure 4: Proof of lower bound
Any set of \((\ln n)^{9/10}\) strips by the above upperbound have total area at most \(\frac{2 \ln n^{9/10}}{n}\). Thus the area of any set of \((\ln n)^{9/10}\) strips divided by the area of \(S\) is at most \(\frac{2}{(\ln n)^{1/10}}\), which is the probability that any given mote in \(S\) falls in the \((\ln n)^{9/10}\) strips.

Let \(p_1\) be the probability that in any small square, a set of at least \(\frac{d}{B} a_n\) motes falls in at most \((\ln n)^{9/10}\) strips. W.h.p. no small square has more than \(3 \ln n\) motes, thus the number of ways to choose a set of \(\frac{d}{B} a_n\) motes from \(3 \ln n\) motes is

\[
\frac{3 \ln n}{\frac{d}{B} a_n} < n^3.
\]

Moreover, as there are \(n/\ln n\) small squares and at most \(n\) ways to choose \((\ln n)^{9/10}\) strips out of \(\ln n\) strips, by Boole’s inequality,

\[
p_1 \leq n^3 \left( \frac{2}{(\ln n)^{1/10}} \right)^{\frac{d \ln n}{\frac{d}{B} a_n}} \leq n^6 \left( \frac{1}{e} \frac{\ln(n) d a_n}{\frac{d}{B} a_n} \right) = n^6 e^{-\frac{2 \ln n}{\ln 9}}.
\]

Therefore, as \(n \to \infty\), a sufficiently large constant \(d\) can be chosen such that \(p_1 \to 0\).

Given the above partition of \(S\) in \(\ln n\) strips, we ignore the first \(\sqrt{\ln n}\) and last \(\sqrt{\ln n}\) strips (keeping the middle strips of larger area). Every strip by construction will either have height or width \(\Theta(1/b_n)\). Without loss of generality assume the orientation of the strips is such that the width is \(\Theta(1/b_n)\). Define the average height of a strip as the average of the two sides of the strip. Consider the worst case, when the difference in height between both sides of a strip is maximal, i.e. the case where the orientation of the partition is either \(\pi/4\) or \(3\pi/4\). Notice that the average height of all middle strips will be larger than the average height of any of the first \(\sqrt{\ln n}\) strips (strip \(T_i\) in Figure 5). Draw a diagonal line \(L\) of length \(\sqrt{\frac{n}{\ln n}}\), \(L\) spans \(\sqrt{\ln n}\) of the discarded strips. The triangle with sides \(L\), \(L'/2\) and the edge of \(S\) is rectangle with two angles of \(\pi/4\), so \(L' = \Theta(\sqrt{\frac{\ln n}{n \ln n}}) = \Theta(\frac{1}{\sqrt{n}})\). In the same way, considering the triangle formed by \(L + \Theta(1/b_n)\) and \(L''/2 = \Theta(1/2\sqrt{n})\) together with the side of \(S\), the average height of strip \(T_i\) is \(\Theta(\frac{1}{\sqrt{n}})\), and the area of any middle strip is at least the area of \(T_i\), which is \(\Theta(\frac{1}{\sqrt{n} \ln n} \times \frac{1}{\sqrt{n}}) = \Theta(\frac{1}{n \sqrt{\ln n}})\).

Using the same arguments used in the proof of lemma 3, we can find a sufficiently large constant \(d\) such that w.h.p., at least \(\frac{d}{\sqrt{B}} a_n\) motes will fall outside of the first and last \(\sqrt{\ln n}\) strips. Consider only these \(\frac{d}{\sqrt{B}} a_n\) motes and label the motes along the specified direction, in the following way: Scan an imaginary line along the orientation of the strips through the \(\ln - 2 \sqrt{\ln n}\) strips. Label the motes from \(m_1\) to \(m_\frac{d}{\sqrt{B}} a_n\), according to the order they are scanned by the line. Partition the motes into disjoint pairs of consecutive motes; motes \(m_{2i-1}\) and \(m_{2i}\) form a pair. For each pair of motes, each of the two motes could be in the same strip or in different strips. Since we have \(\Theta(\ln n)\) strips and \(\frac{d}{\sqrt{B}} a_n\) mote pairs, by the pigeon hole principle, going along the specified direction, for \(d\) chosen sufficiently large in at least \(\frac{d}{8B} a_n\) motes pairs, the two motes in the pair, will be within \(2 \ln n\) strips of each other. Now in order for these motes to be part of the same clique, in each pair both motes must see each other. Without loss of the generality assume the orientation of the strips is parallel to the side of \(S\), such that every mote can see every other mote to its right. Thus at least one of the necessary arcs is present. For the other arc to be present the right-most mote (in the
mote pair) must see the mote to its left. Since the strips in question have a width of at least \( \Theta \left( \frac{1}{\sqrt{n}} \right) \), the horizontal coordinates of both points are drawn u.a.r. from \( \left( 0, \Theta \left( \frac{1}{\sqrt{n}} \right) \right) \). Thus in order to compute the probability of this event we will consider two disjoint cases.

**Case one**, the horizontal coordinates of at least one mote is in the interval \( \left( 0, \frac{1}{\ln n} \right) \). In that case, with probability one the right mote see’s the left mote. The probability of case one occurring is \( \Theta \left( \frac{1}{(\ln n)^{1/10}} \right) \).

**Case two**, the horizontal coordinates of both motes is \( > \frac{1}{(\ln n)^{1/10}} \). In this case since \( c^* \) is a constant, the maximum area a mote see’s of any strip which is within \( 2 \ln \ln n \) strips of it (in a specified direction) is at most \( \Theta(1/(na_n)) \). This follows, since the region of any one strip the mote see’s has at most a width of \( \Theta(r_n/a_n) \) and height \( \Theta(1/b_n) \) (given the strip in question is within \( 2 \ln \ln n \) strips of the mote). Now the left-most mote (in the pair) must fall in a strip. Also (since we are conditioning on being in case two), its horizontal coordinate is drawn u.a.r. from an interval of length \( \frac{1}{(\ln n)^{1/10}} \). Thus the probability the right mote see’s the left mote, conditioned on case two occurring, is at most \( \Theta \left( \frac{1}{(\ln n)^{4/10}} \right) \). Thus the probability the right mote see’s the left mote is at most,

\[
\Theta \left( \frac{1}{(\ln n)^{1/10}} \right) + \Theta \left( \frac{1}{(\ln n)^{4/10}} \right) = \Theta \left( \frac{1}{(\ln n)^{1/10}} \right) .
\]

Since for every pair of motes these events are independent of each other (because only disjoint pairs are being considered), the probability for every pair of motes, both motes see each other is

\[
\leq \Theta \left( \frac{1}{(\ln n)^{1/10}} \right)^{d\ln n / 8B \ln \ln n} .
\]

Let \( p_2 \) denote the probability in any square \( S \) in \( S \), there is a clique of size \( d \cdot a_n \) or greater. Since (w.h.p.) in any square \( S \) we have at most \( n^3 \) sets of size \( d \cdot a_n \) or greater, and \( n/\ln n \) squares in \( S \), by Boole’s inequality

\[
p_2 \leq n^4 \Theta \left( \frac{1}{(\ln n)^{1/10}} \right)^{d\ln n / 8B \ln \ln n} \approx n^4 e^{-d \ln n / 80B} .
\]

Therefore, there is a sufficiently large constant \( d \) such that \( p_2 \to 0 \), and thus w.h.p., \( \omega^* \leq \Theta(a_n) \).

\[\Box\]

**4.3 \( \alpha = 2\pi \)**

For \( \alpha = 2\pi \), a random sector graph is equivalent to a random disk graph. For a random disk graph it is already known w.h.p., \( \omega(G_n) \) is \( \Theta(\ln n) \) [Pen03]. However, this fact can be directly verified, as above, by partitioning the \([0,1]^2\) unit square into \( \epsilon_{2\pi}^n \) regions, and bounding (w.h.p.) the number of motes in any region. Thus the value of \( \omega(G_n) \), for the particular value of \( r_n \) considered in this paper, exhibits two transitions, one at \( \pi + \epsilon \), the other at \( 2\pi \).
5 Proof of Theorem 2

5.1 $\alpha > \pi + \epsilon$

Proof Partition the unit square into $2n/\ln n$, $\left[\frac{\sqrt{\alpha}}{\sqrt{2}} \times \frac{\sqrt{\alpha}}{\sqrt{2}}\right]$ small squares, call this a $S^*$ partition. Observe that all the motes in any small square are at most a distance of $r_n$ apart. Since there are $2n/\ln n$ squares and $n$ motes, by the pigeon hole principle at least one of the small squares has $\ln n/2$ motes. Consider this square which is a subgraph $H$ of $G_n$. For each mote $i$ in $H$ with sector $S_i$, consider the sector $S_i^* = 2\pi - S_i$. It has an amplitude of $\pi - \epsilon$ (see Figure 6). That is, the sector which each mote does not see, equals $\pi - \epsilon$. The motes of any independent set in $H$ must form a clique in $H^*$, where $H^*$ is the sector graph induced by $S_i^*$. Since the amplitude is less than $\pi$, this set must form a convex polygon (as was the case for the clique of $G_n$ when $\alpha < \pi - \epsilon$), thus $w(H^*) \leq \left\lfloor \frac{2\pi}{\alpha - \pi} \right\rfloor$. Let $\vartheta(H)$ represent the independence number of $H$. Then $\vartheta(H) = w(H^*) \leq \left\lfloor \frac{2\pi}{\alpha - \pi} \right\rfloor$. Using the fact that $\chi(G_n) \geq \chi(H) \geq \frac{V_H}{\vartheta(H)}$, we have $\chi(G_n) \geq \frac{\ln n}{2\left(\frac{2\pi}{\alpha - \pi}\right)}$. In order to establish the upper bound we use Brook’s Theorem (see Lemma 1.3 in [MR00]): $\chi(G_n) \leq \Delta(G_n) + 1$. Form [DPS03], we know w.h.p., $\Delta(G_n)$ is $\Theta(\ln n)$. Thus, w.h.p. $\chi(G_n)$ is $\Theta(\ln n)$. \hfill $\Box$

5.2 $\epsilon < \alpha < \pi - \epsilon$

Proof

- First we establish the lower bound. Note that $\tilde{\omega}_2(G_n) \leq \chi(G_n)$, where $\tilde{\omega}_2(G_n)$ is the size of the maximum undirected clique. Consider the dissection $S$ of $[0,1]^2$. Divide each
Figure 6: Sector $S_i$ and complementary sector $S_i^*$

![Diagram of sector and complementary sector]

Figure 7: Figure for the proof of 5.2

Square into $\ln n$ equal regions by placing $\ln n$ equally spaced horizontal lines, i.e. a horizontal subdivision of $S$ (see Figure 3 (b)). By Lemma 2, w.h.p. there is a region $R_i$ which contains $(1 - o(1))a_n/B$ motes from each class in $B$. Consider the motes in the region $R_i$ which belong to class $B_1$. Subdivide $R_i$ into $a_n/B$ rectangles, by drawing $a_n/B$ evenly spaced vertical lines. Thus each rectangle has height equal to $1/b_n$ and width equal to $Br_n/a_n$ (see Figure 7). The expected number of rectangles containing at least one vertex in the limit as $n \to \infty$ is $(1 - \frac{1}{e})a_n/B$; again one can show concentration around this expectation via Talagrand’s inequality, thus w.h.p. there at least $(1 - (\frac{1}{e} + o(1)))a_n/B$ such rectangles. Assume (the worst case) a mote $i$ is in the upper (or lower) right-hand corner of a rectangle, and its orientation angle $\beta_i = 0$, after a distance of $\left\lceil \frac{\cos(\alpha/2)}{\sin(\alpha/2)b_n} \right\rceil$ in the horizontal direction, the mote will be able to see a distance of $1/b_n$ in the vertical direction. That is after this distance the mote will have an arc with every other mote to its right within a distance of $r_n$ in the rectangle in question. Repeating the same arguments as in the case of $\omega(G_n)$ (which we omit in the interest of space), one can establish, w.h.p., $\overline{\omega}_2(G_n)$ is at least $d \cdot a_n$, where $d$ is some constant dependent on $\alpha$.

- Next we establish the upper bound. Consider the dissection $S$ on $[0,1]^2$. Let $\chi^*$ represent the largest chromatic number of any square $S$ in $S$, then it is easily verifiable $\chi(G_n)$ is upper-bounded by $9\chi^*$, thus in order to upperbound $\chi(G_n)$, we upperbound $\chi^*$.

Again fix a square $S$ in $S$ and consider the partition $B$. By the Pigeon hole principle, at least one class $B_j$ will contain $\frac{d}{n}a_n$ motes in $S$ all them oriented in almost the same direction.
Let $M_j$ be the set of all such motes. Define a partition of $S$ into $\ln n$ strips in the following way. Imagine there exists a mote in $S$, whose bisectrix falls exactly in the center of the class $B_j$. Draw a line perpendicular to the bisectrix of this (imaginary) mote. This perpendicular line will be the orientation of the strips (note in the case of $\omega(G_n)$, Theorem 1, a different orientation was used). Partition $S$ into $\ln n$ strips parallel to the orientation (see Figure 8), in this partition of $S$, all the motes in $M_j$ look in the same approximate direction. We wish to prove that for a sufficiently large constant $d$, w.h.p. every set of $\frac{d}{2B} a_n$ motes contains an independent set of size at least $\frac{1}{3} \ln \ln n$.

Using similar arguments as in Section 1, one can show w.h.p. at least $\frac{d}{2B} a_n$ motes fall into a strip having average height $> \Theta \left( \frac{1}{\sqrt{n}} \right)$. Thus we only consider motes falling into the strips having average height $> \Theta \left( \frac{1}{\sqrt{n}} \right)$. Next, we will order these motes going along the specified direction. For example assume the specified direction is going left to right. Then we label the leftmost mote, 1, the second leftmost mote, 2, and so on. Next we partition the $\frac{d}{2B} a_n$ motes into $\frac{d a_n}{2B \ln \ln n}$ classes. Each class $C$ will contain $\ln \ln n$ motes. Again imagine that we are going from left to right, then class one will contain mote1 to mote$_{\ln \ln n}$, and class two will contain the next $\ln \ln n$ motes. Now again by the pigeon hole principle at least one half of these classes occupy at most $2 \ln n \frac{2B \ln \ln n}{d a_n}$ strips. For $d$ sufficiently large this means at least $\frac{d a_n}{4B \ln \ln n}$ classes occupy at most $2 \ln \ln n$ strips.

Now we consider one class of these motes, say class one. We define two edges to be independent of each other if they have no endpoints in common. Thus the edges $a-b$ and $b-c$ are not independent, whereas the edges $a-b$ and $c-d$ are independent (where $a$, $b$, $c$, and $d$ are vertices).
Lemma 4 For any class \( C \) of \( \ln \ln n \) motes, if the largest independent edge set is less than \( 1/3 \ln \ln n \), then there exists an independent set of size \( 1/3 \ln \ln n \) or greater.

Proof of Lemma 4 This follows from the fact that the size of the vertex cover is at most 2 times the size of the maximal independent edge set with minimum cardinality. More specifically, assume the largest set of independent edges in \( C \) is \( \leq 1/3 \ln \ln n \). Remove all the endpoints (along with any of their edges) from the graph. Since this was the largest independent set of edges (i.e. it is trivially maximal), any other edge not in this set must be dependent relative to some edge in this set (otherwise we would have included it in the set). Thus by removing all the endpoints in this set (the one with the most independent edges) we have deleted all the edges in this subgraph (i.e. in the class in question). Each independent edge has two endpoints. The largest such set is by assumption at most \( 1/3 \ln \ln n \). Thus we have removed at most \( 2/3 \ln \ln n \) motes (i.e. vertices). The class to begin with had \( \ln \ln n \) motes. Thus we are left with at least \( 1/3 \ln \ln n \) motes. Also all the edges have been removed, hence these \( 1/3 \ln \ln n \) motes form an independent set and the lemma is proved.

Define \( p_3 \) to be the probability in any one particular class an independent edge set of size \( 1/3 \ln \ln n \) or greater exists. First we will restrict ourselves to classes \( C' \) which occupy \( 2 \ln \ln n \) strips or less (at least half of the classes are of this type). Thus every mote is within \( 2 \ln \ln n \) strips of any other mote in the class. In order for two motes to share an edge, one mote must see the other going along the specified (in our case going from left to right) direction. There are \( \left( \frac{\ln \ln n}{2} \right) \) possible total edges in the class. Recalling the strips have average height of at least \( \Theta \left( \frac{1}{\sqrt{n}} \right) \), the probability any one edge is present is \( < \Theta \left( \frac{\ln \ln n}{(\ln n)^{1/2}} \right) \). The probability any edge exists is independent of any other edge existing (since we are considering independent edge sets). The cardinality of the largest independent edge set is \( \leq 1/2 \ln \ln n \), thus the total number of ways to chose an independent set greater than \( 1/3 \ln \ln n \) is \( < \ln \ln n \ln \ln n \). Hence by Boole’s inequality for \( d \) sufficiently large,

\[
p_3 \leq \Theta \left( \ln \ln n \ln \ln n \left( \frac{\ln \ln n}{(\ln n)^{1/2}} \right)^{1/3 \ln \ln n} \right) \leq \frac{1}{(\ln n)^{4/10}} \equiv e^{-\frac{d \ln n}{40B}}.
\]

Let \( p_4 \) denote the probability in any small square a set with \( d \cdot a_n \) motes does not have an independent set of size \( 1/3 \ln \ln n \) or greater. There are at most \( (w.h.p.) \) \( n^3 \) ways to choose a set of \( d \cdot a_n \) motes in any small square. We are considering \( \frac{d a_n}{4 ln \ln n} \) classes of motes, i.e. all the classes which occupy at most \( 2 \ln \ln n \) strips. No two classes have any motes in common, thus they are independent of each other. And we have \( n/\ln n \) small squares, so for \( d \) sufficiently large, by Boole’s inequality

\[
p_4 \leq n^4 e^{-\frac{(1/10)(\ln \ln n)^2}{\frac{d a_n}{4 ln \ln n}}} \equiv n^4 e^{-\frac{d \ln n}{40B}}.
\]

Therefore, there exists a sufficiently large constant \( d \) such that \( p_4 \to 0 \).

Thus w.h.p., in every small square any set of \( d \cdot a_n \) motes, has an independent set of size at least \( 1/3 \ln \ln n \). Now take any small square, and keep on choosing independent sets of size \( 1/3 \ln \ln n \). Assign all the motes in the same independent set the same color. When there are less than \( d \cdot a_n \) motes left, assign all the remaining motes a different color. Thus we have colored all the motes in any small square (w.h.p.) with at most \( \frac{1}{1/3 \ln \ln n} + d \cdot a_n \) colors. Since the chromatic number of the graph \( \chi(G_n) \), is at most a constant times this amount, w.h.p.

\[
\chi(G_n) = O \left( \frac{\ln n}{\ln \ln n} \right).
\]

Combining with our lower bound, we have \( \chi(G_n) \) is, w.h.p., \( \Theta \left( \frac{\ln n}{\ln \ln n} \right) \).
6 Proof of Theorem 3

6.1 $\alpha > \pi + \epsilon$

Proof First we prove for $\alpha > \pi + \epsilon$, w.h.p., $\hat{\omega}_2 \geq \Theta(\ln n)$. Again, consider the dissection $S$ of $[0, 1]^2$. By the pigeon hole principle at least one of the squares has $\ln n/2$ motes. Further all the motes in this square are at most a distance of $r_n$ apart. Consider the subgraph $H$ induced by the motes in this square $S$ and consider the partition $B$. The expected number of motes in any class $B_i \in B$ is $\frac{\ln n}{2B_i}$. By Lemma 3, w.h.p. every class contains $(1 - o(1))\frac{\ln n}{2B_i}$ motes. Next Divide $S$ into $\frac{\ln n}{2B_i}$ stripes, by drawing $\frac{\ln n}{2B_i}$ evenly spaced vertical lines. The expected number of strips containing at least one vertex in the limit as $n \to \infty$ is $(1 - \frac{1}{e})\frac{\ln n}{2B_i}$; and by Talagrand’s inequality w.h.p., there are at least $(1 - (\frac{1}{e} + o(1)))\frac{\ln n}{2B_i}$ such strips. Consider the motes in $B_1$, going from left to right, every mote can see every other mote to its right (since $\alpha > \pi$). Thus, w.h.p. $\hat{\omega}_2$ is at least $(1 - (\frac{1}{e} + o(1)))\frac{\ln n}{2B_i}$.

For the upper bound, we know w.h.p., $\Delta(G_n) < \Theta(\ln n)$. \hfill \Box

6.2 $\alpha < \pi - \epsilon$

Proof We already established in our proof of $\chi(G_n)$ (see 5.2) a lower bound of $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$. For the upper bound, the proof is similar to the proof of $\omega(G_n)$ (see 4.2), and it is omitted, thus w.h.p., $\hat{\omega}_2 \leq \Theta\left(\frac{\ln n}{\ln \ln n}\right)$. \hfill \Box

7 Conclusions and open problems

In this work, we determined asymptotic values for the directed clique $\omega^*(G_n)$, the modified clique $\hat{\omega}_2(G_n)$ and the chromatic number $\chi(G_n)$ of random scaled sector graphs. We observed $\omega^*$ exhibits a threshold at $\alpha = 2\pi$ and at $\alpha = \pi$, but we have been unable to compute the value of $\omega^*$ for the particular value of $\alpha = \pi$. Similarly, there are thresholds in the behavior of $\hat{\omega}_2(G_n)$ and $\chi(G_n)$ at $\alpha = \pi$. Again, our methods do not seem to work for computing $\hat{\omega}_2(G_n)$ and $\chi(G_n)$, for the particular value of $\alpha = \pi$ and the computation of $\hat{\omega}_2(G_n)$, $\chi(G_n)$ and $\omega^*(G_n)$ at $\alpha = \pi$ remain open problems.

References


