The Price of Anarchy for Polynomial Social Cost

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Abstract

In this work, we consider an interesting variant of the well studied KP model for selfish routing that reflects some influence from the much older Wardrop model [21]. In the new model, user traffics are still unsplittable and links are identical. Social Cost is now the expectation of the sum, over all links, of Latency Costs; each Latency Cost is modeled as a certain polynomial latency cost function evaluated at the latency incurred by all users choosing the link. The resulting Social Cost is called Polynomial Social Cost, or Monomial Social Cost when the latency cost function is a monomial. All considered polynomials are of degree $d$, where $d \geq 1$, and have non-negative coefficients.

We are interested in evaluating Nash equilibria in this model, and we use the Monomial Price of Anarchy and the Polynomial Price of Anarchy as our evaluation measures. Through some unexpected relations of these costs and measures to some classical combinatorial numbers such as the Stirling numbers of the second kind and the Bell numbers, we obtain a multitude of results:

- For the special case of identical users and just two links:
  - The fully mixed Nash equilibrium, where all probabilities are strictly positive, maximizes Polynomial Social Cost.
  - The Monomial Price of Anarchy is exactly $\frac{1}{2}(2^d - 1) + 1$, while the Polynomial Price of Anarchy is no more than $\frac{1}{2}(2^d + d - 1)$.

- For the special case of identical users and an arbitrary number of links:
  - The Polynomial Social Cost is no more than $\left(1 + \frac{1}{n}\right)^{d-1}$ times the Polynomial Social Cost of the fully mixed Nash equilibrium.
  - The Monomial Price of Anarchy is no more than $\left(1 + \frac{1}{n}\right)^{d-1} \cdot B_d$, where $B_d$ denotes the Bell number of order $d$. This immediately implies that the Polynomial Price of Anarchy is no more than $\sum_{1 \leq t \leq d} \left(1 + \frac{1}{n}\right)^{t-1} \cdot B_t$.

- The Monomial Price of Anarchy is exactly $\frac{(2^d - 1)^d}{(d-1)(2^d - 2)^{d-1}} \left(\frac{d-1}{d}\right)^d$ for pure Nash equilibria. This immediately implies that the Polynomial Price of Anarchy is no more than $\sum_{2 \leq t \leq d} \frac{(2^d - 1)^t}{(t-1)(2^d - 2)^{t-1}} \left(\frac{t-1}{t}\right)^t$. 

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1 Introduction

1.1 Framework

The Price of Anarchy [12, 18], also known as Coordination Ratio, is a widely adopted measure of the extent to which competition approximates cooperation. In the most general setting, the Price of Anarchy is the worst-case ratio between the value of a global objective function called Social Cost [12] over its optimal value, called Optimum. The Social Cost is evaluated at a Nash equilibrium [16, 17]; here, no user could unilaterally switch from its own (mixed) strategy in order to improve the value of its local objective function, called (Expected) Individual Cost. Yet, Optimum is the solution to some, usually hard, combinatorial optimization problem. So, the Price of Anarchy represents a rendezvous of Nash equilibrium, a fundamental concept from Game Theory, with approximation, an ubiquitous concept from Theoretical Computer Science.

Koutsoupias and Papadimitriou [12] introduced the Price of Anarchy in the context of some specific setting, widely known as the KP model and extensively studied in the last few years as a prevailing model for selfish routing (see, e.g., [3, 4, 6, 8, 9, 11, 14, 15]). In the KP model, there are $m$ parallel links and $n$ selfish users with (unsplittable) traffics. The (expected) latency incurred on a link is the (expected) total traffic of users choosing it. The (Expected) Individual Cost of a user is the (expected) latency on the link it chooses. In a Nash equilibrium, each user alone is minimizing its (Expected) Individual Cost. The Social Cost is the expectation of maximum latency; the Optimum is the least possible maximum latency.

The Wardrop model [21] is another prevailing model for selfish routing that dates back to the 1950s, when it was considered in the context of road traffic networks. In the Wardrop model, the network can be arbitrary; user traffics are infinitesimally splittable, and this rules out mixed strategies from consideration. In addition, Social Cost is defined here as the sum of all Individual Costs; each Individual Cost is a certain sum of Latency Costs. More specifically, the Latency Cost on a link is determined by a convex function, called latency cost function, of the latency on the link; the Individual Cost of a user is the sum of Latency Costs on links in the paths chosen by the user. Inspired by the vivid interest in the Price of Anarchy, Roughgarden and Tardos [19] initiated recently a reinvestigation of the Wardrop model.

1.2 The Model and its Relatives

A natural goal is to understand the dependence of the Price of Anarchy on the particular way of formulating Individual Cost and Social Cost. Towards this goal, some recent works [7, 13] have considered bridging the KP model with the Wardrop model and analyzing the bridged model. In this paper, we further pursue this goal by introducing and analyzing a new, interesting variant of the KP model that reflects some influence from the Wardrop model.
In our proposed model, we follow the KP model to consider the parallel links network, unsplittable traffics and mixed strategies. However, inspired by the Wardrop model, we introduce Polynomial Social Cost as the expectation of the sum of Latency Costs on links. We also assume that latency cost functions are polynomial; all polynomials we consider are of degree \( d \) and have non-negative coefficients. We assume that links are identical; so, all polynomials are identical. Polynomial Social Cost gives rise to Polynomial Optimum and Polynomial Price of Anarchy in the natural way. In some cases, we also consider monomials (that is, polynomials consisting of a single power with unit coefficient). We then talk about Monomial Social Cost, Monomial Optimum and Monomial Price of Anarchy.

Our model is closely related to two previously studied models:

- Relaxed to allow arbitrary links with (not necessarily identical) linear latencies, but restricted to quadratic latency cost functions, our model has been already studied by Lücking et al. [13]. The model in [13] adopted Quadratic Social Cost, which it defined (in an equivalent way) as the sum of weighted (Expected) Individual Costs. Quadratic Social Cost is the special case of our Monomial Social Cost with monomials of degree 2.

- Relaxed to allow arbitrary links with (not necessarily identical) convex latencies, our model was already studied by Gairing et al. [7]. However, Gairing et al. [7] modeled Social Cost as the sum of Individual Costs. Assuming identical users and linear latencies, the model of Gairing et al. [7] and the model of Lücking et al. [13] become identical.

The relation of our model to the KP model and the models of Lücking et al. [13] and Gairing et al. [7] is summarized in Figure 1. Restricted to pure Nash equilibria, where each user chooses a single link (with probability 1), our model was already studied for monotone latency cost functions in [2]. Besides pure Nash equilibria, we shall pay, in our study, some particular attention to the fully mixed Nash equilibrium [15], where each user chooses each link with non-zero probability.

### 1.3 Contribution and Significance

We are primarily interested in analyzing the Polynomial Price of Anarchy for our new model. To do so, we introduce and study a natural conjecture, called the Polynomial Fully Mixed Nash Equilibrium Conjecture and abbreviated as the PFMNE Conjecture; it asserts that the fully mixed Nash equilibrium maximizes Polynomial Social Cost. Although the PFMNE conjecture is interesting in its own right, a resolution of it to the positive would also enable the derivation of upper bounds on the Polynomial Price of Anarchy via deriving upper bounds on the Polynomial Social Cost of the fully mixed Nash equilibrium.

We address two important special settings of the problem:
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Figure 1: Comparison of present model to three relatives (namely, the KP model [12] and the models in [7, 13]). Note that all four models formulate Individual Cost (as a function of latency) in the same way, while they do not (in general) do so for latency. In the special case where the linear function for latency in the KP model and the model of Lücking et al. [13] becomes the identity function, the Individual Costs for these models become identical with the one for the present model, as also do their Nash equilibria.

1.3.1 Identical Users

For the case of identical users, we rely on a very thorough analysis of the fully mixed Nash equilibrium. For the analysis, we employ, as our chief combinatorial instrument, the binomial function originally introduced in [7]. We prove here that the binomial function can be expressed as a combinatorial sum of Stirling numbers of the second kind [20] (Proposition 2.2). We also observe that Polynomial Social Cost can be expressed as a sum of binomial functions (in the case of identical users). Together these two imply that the Polynomial Social Cost of the fully mixed Nash equilibrium is a combinatorial sum of Stirling numbers of the second kind. Moreover, the Polynomial Social Cost of any (mixed) Nash equilibrium is upper bounded by a certain combinatorial sum of Stirling numbers of the second kind. Hence, comparison of these two Polynomial Social Costs reduces to comparing like terms in the two combinatorial sums.

Two Links

We obtain the following results:

- The PFMNE Conjecture is valid. The proof follows a careful comparison of like terms in the combinatorial sums expressing the Polynomial Social Costs of an arbitrary and the fully mixed Nash equilibria.
• The Monomial Price of Anarchy is exactly $\frac{1}{2}(2^d - 1)$ (Theorem 4.3). From this, an upper bound of $\frac{1}{2}(2^d + d - 1)$ on Polynomial Price of Anarchy follows immediately (Corollary 4.4).

Many Links

We extend the techniques we used for the case of two links to prove:

• The Monomial Social Cost of any Nash equilibrium is no more than $(1 + \frac{1}{n})^{d-1}$ times the Monomial Social Cost of the fully mixed Nash equilibrium (Theorem 5.1). This result proves an approximate version of the PFMNE Conjecture; in fact, we believe that the factor $(1 + \frac{1}{n})^{d-1}$ is not necessary.

• The Monomial Price of Anarchy is upper bounded by $(1 + \frac{1}{n})^{d-1} \cdot B_d$, (Theorem 5.4); here, $B_d$ is the Bell number of order $d$, which already exceeds the corresponding (tight) bound of $\frac{1}{2}(2^d - 1)$ for two links. This follows from the shown approximate version of the PFMNE Conjecture; so, we believe that here also the factor $(1 + \frac{1}{n})^{d-1}$ is not necessary. From this bound, an upper bound of $\sum_{1\leq t \leq d} (1 + \frac{1}{n})^{t-1} \cdot B_t$ on Polynomial Price of Anarchy follows immediately (Corollary 5.5).

1.3.2 Pure Nash Equilibria

The Monomial Price of Anarchy is exactly $\frac{(2^d - 1)^d}{(d-1)(2^d - 2)^{d-1}} \left( \frac{d-1}{d} \right)^d$ (Theorem 6.4). From this, an upper bound of $\sum_{2\leq t \leq d} \frac{(2^t - 1)^t}{(t-1)(2^t - 2)^{t-1}} \left( \frac{t-1}{t} \right)^t$ on Polynomial Price of Anarchy follows immediately (Corollary 6.5).

1.3.3 Summary and Remarks

All shown bounds are summarized in Figure 2. We remark that all shown bounds are independent of $m$ and $n$, but depend on $d$. The lower bounds imply that this dependence is inherent. Finally, we remark that all upper bounds on Polynomial Price of Anarchy are obtained through naive reductions to corresponding upper bounds on Monomial Price of Anarchy. At present, we do not know if there are better bounds on Polynomial Price of Anarchy that bypass the naive reduction.
1.4 Related Work and Comparison

Gairing et al. [8, 9] were the first to explicitly state the related *Fully Mixed Nash Equilibrium Conjecture* that the fully mixed Nash equilibrium maximizes Social Cost for the KP model. Up to now, the conjecture has been proved for several particular cases of the KP model [6, 8, 14]. Recently, Fischer and Vöcking [5] presented a counterexample to the Fully Mixed Nash Equilibrium Conjecture for the case of identical links. The validity of the Fully Mixed Nash Equilibrium Conjecture for the case of identical users (but arbitrary links) remains open.

Lücking et al. [13] formulated the *Quadratic Fully Mixed Nash Equilibrium Conjecture*, which asserts that the fully mixed Nash equilibrium maximizes Quadratic Social Cost for their model. Lücking et al. [13, Theorem 4.8] proved the Quadratic Fully Mixed Nash Equilibrium Conjecture for the case of identical users and identical links. Our PFMNE Conjecture generalizes the Quadratic Fully Mixed Nash Equilibrium Conjecture of Lücking et al. [13] to polynomial latency cost functions of arbitrary degree. Gairing et al. [7] also formulated a corresponding conjecture for their model, stating that the fully mixed Nash equilibrium maximizes Social Cost (sum of Individual Costs) for their model. Gairing et al. [7, Theorem 3.5] proved their conjecture for the case of identical users and arbitrary links with non-decreasing, non-constant and convex latencies.

Our exact bound on Monomial Price of Anarchy for pure Nash equilibria includes, as the special case with $d = 2$, the exact bound of $\frac{9}{8}$ on Quadratic Price of Anarchy for pure Nash equilibria shown in [13, Theorem 5.2]. Our proof generalizes the one for [13, Theorem 5.2].

Our exact bound of $\frac{1}{2}(2^{d-1} + 1)$ on Monomial Price of Anarchy for the case of identical users and two (identical) links implies, as the special case where $d = 2$, a tight bound of $\frac{3}{2}$.
on Quadratic Price of Anarchy. This complements the corresponding exact bound of $\frac{4}{3}$ shown in [13, Theorem 5.1] for the case of identical users and pure Nash equilibria (but for arbitrarily many links).

Our upper bound of $\left(1 + \frac{1}{n}\right)^{d-1} \cdot B_d$ for the case of identical users and many links implies, as the special case where $d = 2$, an upper bound of $\left(1 + \frac{1}{n}\right)^{\frac{1}{2}} \cdot B_2 = 2 \left(1 + \frac{1}{n}\right) > 2$. The implied upper bound exceeds the corresponding upper bound of $1 + \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\} < 2$ shown in [13, Theorem 5.4] for the same case. So, our upper bound of $\left(1 + \frac{1}{n}\right)^{d-1} \cdot B_d$ for the case of identical users and many links is not tight in the particular case where $d = 2$.

Other bounds on Price of Anarchy include tight asymptotic bounds (depending on $m$) for the KP model [3, 11] and exact constant bounds for the Wardrop model [19].

1.5 Road Map

Section 2 summarizes some mathematical preliminaries. Section 3 introduces our theoretical model. The case of identical users is considered in Sections 4 (for two links) and 5 (for arbitrarily many links). Pure Nash equilibria are treated in Section 6. We conclude, in Section 7, with a discussion of our results and some open problems.

2 Mathematical Preliminaries

Throughout, denote for any integer $k \geq 1$, $[k] = \{1, \cdots, k\}$. A monomial function $g : \mathbb{R} \to \mathbb{R}$ has the form $g(\lambda) = \lambda^d$ for some integer $d \geq 0$. A polynomial function is a linear combination of monomials. We shall only consider polynomial functions with non-negative coefficients. For a random variable $X$ with associated probability distribution $P$, denote $E_P(X)$ the expectation of $X$. In some later proof, we shall make use of the following simple mathematical fact that follows directly from the convexity of the monomial function $\tau_d(\lambda) = \lambda^d$.

Lemma 2.1 Let $x, y_1, y_2 \in \mathbb{R}$ with $0 < x \leq y_1 \leq y_2$. Then, for each integer $d \geq 1$,

$$(y_1 - x)^d + (y_2 + x)^d \geq y_1^d + y_2^d.$$ 

2.1 Falling Factorials, Stirling Numbers and Bell Numbers

For any pair of integers $k \geq 1$ and $t \geq 1$, the falling factorial of $k$ order $t$, denoted as $k^\underline{t}$, is given by $k^\underline{t} = k \cdot (k-1) \cdot \ldots \cdot (k - (t - 1))$. 

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For any pair of integers \( d \geq 1 \) and \( t \in [d] \cup \{0\} \), the \textit{Stirling number of the second kind}, denoted as \( S(d, t) \), counts the number of partitions of a set with \( d \) elements into exactly \( t \) blocks (non-empty subsets). In particular, \( S(d, 1) = 1 \). Also, for all integers \( d \geq 2 \), \( S(d, 2) = 2^{d-1} - 1 \).

Stirling numbers of the second kind satisfy the recurrence relation

\[
S(d, t) = \sum_{q \leq t \leq d-1} \binom{d-1}{q-1} \cdot S(q-1, t-1)
\]

for all integers \( d \geq 2 \) and \( t \in [d] \) (see, e.g., [10, Table 265, Identity (6.15)]). It is also known that for all integers \( d \geq 2 \) and \( k \geq 1 \),

\[
k^d = \sum_{t \in [d]} S(d, t) \cdot k^t.
\]

For any integer \( d \geq 1 \), the \textit{Bell number} of order \( d \) [1], denoted as \( B_d \), counts the number of partitions of a set with \( d \) elements into blocks. So, clearly, \( B_0 = 1 \) and \( B_d = \sum_{t \in [d]} S(d, t) \).

### 2.2 Binomial Function

We start with the definition of a binomial function [7, Definition 1].

**Definition 2.1** For any integer \( r \geq 1 \), consider a probability vector \( \mathbf{p} = \langle p_1, \ldots, p_r \rangle \). Fix a function \( g(\lambda) : \mathbb{R} \to \mathbb{R} \). Then, the binomial function \( BF(\mathbf{p}, g) \) is given by

\[
BF(\mathbf{p}, g) = \sum_{A \subseteq [r]} \left( \prod_{k \in A} p_k \cdot \prod_{k \notin A} (1 - p_k) \cdot g(|A|) \right).
\]

Strictly speaking, Definition 2.1 defines a \textit{functional}. If all probabilities have the same value \( p \), then we (abuse notation to) write \( BF(p, r, g) \). Clearly, in this case,

\[
BF(p, r, g) = \sum_{0 \leq k \leq r} \binom{r}{k} p^k (1-p)^{r-k} g(k).
\]

We show that when \( g \) is monomial, the binomial function takes a special form.

**Proposition 2.2** For each integer \( d \geq 1 \),

\[
BF(p, r, \lambda^d) = \sum_{t \in [d]} p^t \cdot S(d, t) \cdot \lambda^t.
\]

**Proof:** By induction on \( r \). For the basis case, let \( r = 1 \). Then, \( BF(p, 1, \lambda^d) = \binom{1}{1} p^1 1^d = p \) and \( \sum_{t \in [d]} p^t S(d, t) 1^t = p^1 S(d, 1) 1^1 = p \), so that the claim follows.

Assume inductively that for some integer \( r \geq 2 \), for each integer \( d \geq 1 \),

\[
BF(p, r-1, \lambda^d) = \sum_{t \in [d]} p^t \cdot S(d, t) \cdot (r-1)^t.
\]
For the induction step, we derive that

\[ \text{BF}(p, r, \lambda^d) = \sum_{k \in [r]} \binom{r}{k} p^k (1 - p)^{r-k} k^d \]

\[ = \sum_{k \in [r]} \frac{r}{k} \binom{r-1}{k-1} p^k (1 - p)^{r-k} k^d \]

\[ = p \cdot r \cdot \sum_{k \in [r]} \binom{r-1}{k-1} p^k (1 - p)^{r-k} k^{d-1} \]

\[ = p \cdot r \cdot \sum_{0 \leq k \leq r-1} \binom{r-1}{k} p^k (1 - p)^{r-k} k^{d-1} \]

\[ = p \cdot r \cdot \sum_{0 \leq k \leq r-1} \binom{r-1}{k} p^k (1 - p)^{r-k} \left( \sum_{0 \leq q \leq d-1} \binom{d-1}{q} k^q \right) \]

\[ = p \cdot r \cdot \sum_{0 \leq q \leq d-1} \frac{d-1}{q} \text{BF}(p, r - 1, \lambda^q) \]

\[ = p \cdot r \cdot \sum_{0 \leq q \leq d-1} \frac{d-1}{q} \text{BF}(p, r - 1, \lambda^q) + p \cdot r \cdot \sum_{q \in [d-1]} \binom{d-1}{q} \text{BF}(p, r - 1, \lambda^q) \]

\[ = p \cdot r + p \cdot r \cdot \sum_{q \in [d-1]} \binom{d-1}{q} \text{BF}(p, r - 1, \lambda^q) \]

\[ = p \cdot r + p \cdot r \cdot \sum_{q \in [d-1]} \binom{d-1}{q} \left( \sum_{t \in [d]} p^t \cdot S(q, t) \cdot (r - 1)^t \right) \]

\[ = p \cdot r + \sum_{q \in [d-1]} \binom{d-1}{q} \left( \sum_{t \in [d]} p^{t+1} \cdot S(q, t) \cdot r^{t+1} \right) \]

\[ = p \cdot r + \sum_{t \in [d-1]} p^t \cdot r^{t+1} \cdot \left( \sum_{q \leq t \leq d-1} \binom{d-1}{q} \cdot S(q, t) \right) \]

\[ = p \cdot r + \sum_{t \in [d-1]} p^t \cdot r^{t+1} \cdot \left( \sum_{q \leq t \leq d-1} \binom{d-1}{q} \cdot S(q - 1, t - 1) \right) \]

\[ = p \cdot r + \sum_{t \in [d]} p^t \cdot r^t \cdot S(d, t) \]

\[ = \sum_{t \in [d]} p^t \cdot r^t \cdot S(d, t), \]

as needed.
Proposition 2.2 implies that for a constant probability vector and a monomial function, the binomial function is a combinatorial sum of Stirling numbers of the second kind.

It is known [7, Lemma 3] that in case $g$ is convex, the binomial function does not decrease when replacing all probabilities in the probability vector $p$ by the average probability $\tilde{p} = \frac{1}{n} \sum_{i \in [n]} p_i$.

Lemma 2.3 ([7]) For a convex function $g$, $BF(p, g) \leq BF(\tilde{p}, r, g)$.

## 3 Model and Preliminaries

Our model definitions are built on top of those for the KP model [12], which are extended to accommodate some features from the Wardrop model [21].

### 3.1 General

We consider a simple network consisting of a set of $m$ parallel links $1, 2, \ldots, m$ from a source node to a destination node. Each of $n$ users $1, 2, \ldots, n$ wishes to route a traffic along a (non-fixed) link from source to destination. Denote $w_i$ the traffic of user $i \in [n]$; denote $W = \sum_{i \in [n]} w_i$. Define the $n \times 1$ traffic vector $w$ in the natural way. We assume that all links are identical.

Thus, an instance is a tuple $(w, m)$. In the model of identical users, all traffics are equal to 1. In that case, an instance is a pair $(n, m)$. Assume throughout that $m \geq 2$ and $n \geq 2$.

The latency $\lambda$ on a link is the total traffic on it. Associated with each link is a latency cost function, which is a polynomial $\pi_d(\lambda) = \sum_{0 \leq i \leq d} a_i \lambda^i$ of degree $d \geq 1$ with non-negative coefficients. In the special case of a monomial, $\pi_d(\lambda) = \lambda^d$. The Latency Cost for latency $\lambda$ on the link is given by $\pi_d(\lambda)$.

### 3.2 Strategies and Assignments

A pure strategy for user $i \in [n]$ is some specific link. A mixed strategy for user $i \in [n]$ is a probability distribution over pure strategies; so, it is a probability distribution over links.

A pure assignment is an $n$-tuple $L = (\ell_1, \ell_2, \ldots, \ell_n) \in [m]^n$; a mixed assignment is an $n \times m$ probability matrix $P$ of $nm$ probabilities $p_{ij}$, with $i \in [n]$ and $j \in [m]$, where $p_{ij}$ is the probability that user $i$ chooses link $j$. A user $i \in [n]$ is pure in the mixed assignment $P$ if its (mixed) strategy is pure; otherwise, user $i$ is mixed. Clearly, all users are pure in a pure assignment. A mixed assignment $P$ is fully mixed [15, Section 2.2] if for all users $i \in [n]$ and links $j \in [m]$, $p_{ij} > 0$. 

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Fix now a mixed assignment \( \mathbf{P} \). The latency \( \lambda_j(\mathbf{P}) \) on link \( j \in [m] \) induced by \( \mathbf{P} \) is the total traffic assigned to the link according to \( \mathbf{P} \); so, \( \lambda_j(\mathbf{P}) \) is a random variable. Denote \( \Lambda_j(\mathbf{P}) \) the expected latency on link \( j \in [m] \); thus, \( \Lambda_j(\mathbf{P}) = \mathbb{E}_\mathbf{P}(\lambda_j(\mathbf{P})) = \sum_{i \in [n]} p_{ij} w_i \).

### 3.3 Cost Measures

#### 3.3.1 Individual Cost and Expected Individual Cost

For a pure assignment \( \mathbf{L} \), the Individual Cost for user \( i \in [n] \), denoted as \( \text{IC}_i(\mathbf{L}) \), is \( \text{IC}_i(\mathbf{L}) = \Lambda_{i_i}(\mathbf{L}) \); so, the Individual Cost for a user is the latency of the link it chooses. For a mixed assignment \( \mathbf{P} \), the Expected Individual Cost for user \( i \in [n] \), denoted again as \( \text{IC}_i(\mathbf{P}) \), is the expectation according to \( \mathbf{P} \) of the Individual Cost for the user.

The Conditional Expected Individual Cost \( \text{IC}_{ij}(\mathbf{P}) \) for user \( i \in [n] \) on link \( j \in [m] \) is the conditional expectation according to \( \mathbf{P} \) of the Individual Cost of user \( i \) had it been assigned to link \( j \). Clearly, for each user \( i \in [n] \), \( \text{IC}_i(\mathbf{P}) = \sum_{j \in [m]} p_{ij} \text{IC}_{ij}(\mathbf{P}) \).

#### 3.3.2 Polynomial Social Cost

Associated with an instance \( \langle \mathbf{w}, m \rangle \), a latency cost function \( \pi_d(\lambda) \) and a mixed assignment \( \mathbf{P} \) is the Polynomial Social Cost, denoted \( \text{PSC}_{\pi_d(\lambda)}(\mathbf{w}, m, \mathbf{P}) \), which is the expectation of the sum of Latency Costs; so, by linearity of expectation,

\[
\text{PSC}_{\pi_d(\lambda)}(\mathbf{w}, m, \mathbf{P}) = \mathbb{E}_\mathbf{P}\left( \sum_{j \in [m]} \pi_d\left( \sum_{k \in [n] : \ell_k = j} w_k \right) \right)
\]

\[
= \sum_{j \in [m]} \mathbb{E}_\mathbf{P}\left( \pi_d\left( \sum_{k \in [n] : \ell_k = j} w_k \right) \right)
\]

\[
= \sum_{j \in [m]} \sum_{A \subseteq [n]} \left( \prod_{i \in A} p_{ij} \right) \left( \prod_{i \in \bar{A}} (1 - p_{ij}) \right) \pi_d\left( \sum_{k \in [n] : \ell_k = j} w_k \right).
\]

Note that

\[
\text{PSC}_{\pi_d(\lambda)}(\mathbf{w}, m, \mathbf{P}) = \sum_{0 \leq t \leq d} a_t \cdot \text{PSC}_{\lambda^t}(\mathbf{w}, m, \mathbf{P}).
\]

So, Polynomial Social Cost is a linear combination (with non-negative coefficients) of Monomial Social Costs. This property will later reduce the comparison of the Polynomial Social Costs of two different assignments to the pairwise comparison of their Monomial Social Costs.

We remark that the Polynomial Social Cost is a generalization of the Quadratic Social Cost [13] to latency cost functions that are polynomials of arbitrary degree.
3.3.3 Polynomial Optimum

Associated with an instance \((w, m)\) and a latency cost function \(\tau_d(\lambda)\) is the Polynomial Optimum, denoted \(\text{POPT}_{\tau_d}(w, m)\), which is the least possible, over all pure assignments, Polynomial Social Cost; thus,

\[
\text{POPT}_{\tau_d}(w, m) = \min_{L \in [m]^n} \text{PSC}_{\tau_d}(w, m, L).
\]

A (pure) assignment \(L\) such that \(\text{PSC}_{\tau_d}(w, m, L) = \text{POPT}_{\tau_d}(w, m)\) will be called optimal (for the instance \((w, m)\) and the latency cost function \(\tau_d(\lambda)\)). We remark that the Polynomial Optimum is a generalization of the Quadratic Optimum [13] to latency cost functions that are polynomials of arbitrary degree. Monomial Optimum is defined as the natural special case of Polynomial Optimum.

3.4 Nash Equilibria

Given an instance \((w, m)\), the mixed assignment \(P\) is a Nash equilibrium [12, Section 2] if for each user \(i \in [n]\), it minimizes the Expected Individual Cost \(\text{IC}_i(P)\) over all mixed assignments \(Q\) that differ from \(P\) only with respect to the mixed strategy of user \(i\); that is, for all such mixed assignments \(Q\), \(\text{IC}_i(P) \leq \text{IC}_i(Q)\). Thus, in a Nash equilibrium, there is no incentive for a user to unilaterally deviate from its mixed strategy.

We remark that latency and (Expected) Individual Cost are defined for our model in the same way they are defined for the KP model (and for the model of Lücking et al. [13] as well) in the case of identical links. This implies that the sets of Nash equilibria for the two models coincide.

The particular definition of Expected Individual Cost implies that in a Nash equilibrium, for each user \(i \in [n]\) and link \(j \in [m]\) such that \(p_{ij} > 0\), all Conditional Expected Individual Costs \(\text{IC}_{ij}(P)\) are the same and no more than any Conditional Expected Individual Cost \(\text{IC}_{ij}(P)\) with \(p_{ij} = 0\).

In a typical Nash equilibrium \(P\), the probabilities \(\{p_{ij} \mid i \in [n]\}\) for a fixed link \(j \in [m]\) are not necessarily identical. In such case, denote as \(\tilde{p}_j\) the average probability \(\tilde{p}_j = \frac{\sum_{i \in [n]} p_{ij}}{|\{i \in [n] \mid p_{ij} > 0\}|}\) on link \(j \in [m]\).

3.5 The Fully Mixed Nash Equilibrium

For the KP model, it is known [15] that the fully mixed Nash equilibrium \(F\) exists uniquely in the case of identical links (with \(f_{ij} = \frac{1}{m}\) for all users \(i \in [m]\) and links \(j \in [m]\)). As the set of Nash equilibria in the KP model (in the case of identical links) and the present model coincide, the same holds for the fully mixed Nash equilibria \(F\) in our model.
We formulate a natural conjecture related to Polynomial Social Costs of Nash equilibria in our model, called the Polynomial Fully Mixed Nash Equilibrium Conjecture and abbreviated as the PFMNE Conjecture.

**Conjecture 3.1 (Polynomial Fully Mixed Nash Equilibrium Conjecture)** For any instance \((w, m)\) and associated Nash equilibrium \(P\), \(\text{PSC}_{\tau\lambda}(w, m, P) \leq \text{PSC}_{\tau\lambda}(w, m, F)\).

The PFMNE Conjecture generalizes the Quadratic Fully Mixed Nash Equilibrium Conjecture to latency cost functions that are polynomials of arbitrary degree. It is also a variant of the well known Fully Mixed Nash Equilibrium Conjecture [8, 9] for the original KP model.

### 3.6 Monomial and Polynomial Price of Anarchy

The Polynomial Price of Anarchy, denoted \(\text{PPoA}\), is the worst-case ratio \(\frac{\text{PSC}_{\tau\lambda}(w, m, P)}{\text{POPT}_{\tau\lambda}(w, m)}\) over all instances \((w, m)\) and associated Nash equilibria \(P\). This generalizes the Quadratic Price of Anarchy [13] to latency cost functions that are polynomials of arbitrary degree. The Monomial Price of Anarchy, denoted \(\text{MPoA}\), is the natural special case of the Polynomial Price of Anarchy.

The following simple fact will be instrumental for reducing the Polynomial Price of Anarchy for arbitrary polynomials (with non-negative coefficients) to the Monomial Price of Anarchy.

**Lemma 3.1 (From Polynomials to Monomials)** Fix any instance \((w, m)\) with an associated Nash equilibrium \(P\). Then,

\[
\frac{\text{PSC}_{\tau\lambda}(w, m, P)}{\text{POPT}_{\tau\lambda}(w, m)} \leq \sum_{t \in [d]} \frac{\text{PSC}_{\lambda_t}(w, m, P)}{\text{POPT}_{\lambda_t}(w, m)}.
\]

**Proof:** Let \(Q\) be an optimal assignment for the instance \((w, m)\). Then, clearly,

\[
\frac{\text{PSC}_{\tau\lambda}(w, m, P)}{\text{POPT}_{\tau\lambda}(w, m)} = \frac{\text{PSC}_{\tau\lambda}(w, m, P)}{\text{PSC}_{\tau\lambda}(w, m, Q)}
\]

\[
= \frac{a_0 + \sum_{t \in [d]} a_t \cdot \text{PSC}_{\lambda_t}(w, m, P)}{a_0 + \sum_{t \in [d]} a_t \cdot \text{PSC}_{\lambda_t}(w, m, Q)}
\]

\[
\leq \frac{\sum_{t \in [d]} a_t \cdot \text{PSC}_{\lambda_t}(w, m, P)}{\sum_{t \in [d]} a_t \cdot \text{PSC}_{\lambda_t}(w, m, Q)}
\]

\[
\leq \sum_{t \in [d]} \frac{a_t \cdot \text{PSC}_{\lambda_t}(w, m, P)}{\text{POPT}_{\lambda_t}(w, m)}
\]

as needed. \(\square\)
3.7 Identical Users

Restricted to identical users, Polynomial Social Cost reduces to

\[ \text{PSC}_{\pi_d(\lambda)}(n, m, P) = \sum_{j \in [m]} \sum_{A \subseteq [n]} \left( \prod_{i \in A} p_{ij} \right) \left( \prod_{i \notin A} (1 - p_{ij}) \right) \pi_d(|A|) \]
\[ = \sum_{j \in [m]} \text{BF}(\langle p_{1j}, \ldots, p_{nj} \rangle, \pi^d(\lambda)) \]

So, in this case, Polynomial Social Cost is a sum of binomial functions, one for each link. Recall also that in the case of identical users, all probabilities are identical (and equal to \( \frac{1}{m} \)) for the fully mixed Nash equilibrium \( F \). Hence, Proposition 2.2 implies now that the Monomial Social Cost of the fully mixed Nash equilibrium \( F \) is a combinatorial sum of Stirling numbers of the second kind.

**Lemma 3.2** Consider the case of identical users. Fix an instance \( (n, m) \). Then,

\[ \text{PSC}_{\lambda^d}(n, m, F) = m \sum_{t \in [d]} \left( \frac{1}{m} \right)^t \cdot S(d, t) \cdot n^t. \]

A lower bound on Monomial Optimum for the case of identical users is \( \text{POPT}_{\lambda^d}(w, m) \geq m \left( \frac{n}{m} \right)^d \) if \( n \geq m \), and \( \text{POPT}_{\lambda^d}(n, m) \geq n \) if \( n < m \).

4 Identical Users and Two Links

In this section, we consider the case of identical users and two links. The PFMNE Conjecture is considered in Section 4.1. Bounds on the Monomial and Polynomial Prices of Anarchy are presented in Section 4.2.

4.1 The PFMNE Conjecture

We prove:

**Theorem 4.1** For the case of identical users and two links, the PFMNE Conjecture is valid.

**Proof:** Fix an instance \( (n, m) \) with associated Nash equilibrium \( P \) and fully mixed Nash equilibrium \( F \). Since Polynomial Social Cost is a linear combination (with non-negative coefficients) of Monomial Social Costs, it suffices to prove the PFMNE Conjecture for a monomial latency cost function \( \pi_d(\lambda) = \lambda^d \). We partition the set of users \([n]\) into three sets: \( U_1 \) (resp., \( U_2 \)) is the set of pure users assigned to link 1 (resp., link 2); \( U_{12} \) is the set of mixed users. Since links
are identical, we may assume, without loss of generality, that $|U_1| \leq |U_2|$. Denote $u = |U_1|$, $v = |U_2| - u$ and $r = |U_{12}|$. We will treat separately pure and non-pure Nash equilibria. In the second case, we will distinguish between the subcase where there are no pure users assigned to one of the two links (that is, $u = 0$), and the subcase where there are pure users assigned to each of the two links (that is, $u > 0$). The second subcase will be reduced to the first. We now continue with the details of the formal proof. We proceed by case analysis.

1. Assume first that $P$ is pure. Since $P$ is a Nash equilibrium, $\Lambda_1(P) \leq \Lambda_2(P) + 1$ and $\Lambda_2(P) \leq \Lambda_1(P) + 1$. So, $|\Lambda_1(P) - \Lambda_2(P)| \leq 1$ or $|2\Lambda_1(P) - n| \leq 1$. Note that $\text{PSC}_d(n, m, P) = (\Lambda_1(P))^2 + (\Lambda_2(P))^2 = (\Lambda_1(P))^2 + (n - \Lambda_1(P))^2$. Hence, $\text{PSC}_d(n, m, P)$ is minimum when $|2\Lambda_1(P) - n| \leq 1$. It follows that $P$ is an optimal assignment, so that $\text{PSC}_d(n, m, P) \leq \text{PSC}_d(n, m, F)$, as needed.

2. Assume now that $P$ is not pure, so that $r > 0$. There are two separate cases.

(a) Assume first that $u = 0$; so, no pure user is assigned to link 1. We first prove that, in this case, $r > 1$.

Assume, by way of contradiction, that $r = 1$, and consider the single mixed user $i_0$. Then, $IC_{i_0}(P) = 1$, while $IC_{i_0}(P) = |U_2| + 1$. Since $P$ is a Nash equilibrium, $IC_{i_0}(P) = IC_{i_0}(P)$. It follows that $|U_2| = 0$. This implies that $n = r = 1$, a contradiction. It follows that $r > 1$. So, $r \in [n - 1] \setminus \{1\}$ mixed users are assigned to both links and $n - r$ pure users are assigned to link 2.

Consider any arbitrary user $i \in U_{12}$. Clearly, $IC_{i_1}(P) = \Lambda_1(P) - p_1 + 1$ and $IC_{i_2}(P) = \Lambda_2(P) - p_2 + 1 = \Lambda_2(P) - (1 - p_1) + 1$. Since $P$ is a Nash equilibrium, $IC_{i_1}(P) = IC_{i_2}(P)$. This equality implies that $p_1 = \frac{\Lambda_1(P) - \Lambda_2(P) + 1}{2}$. So, $p_1$ is independent of $i$. It follows that each mixed user chooses link 1 with probability $p_1 = p_1$ and link 2 with probability $p_2 = 1 - p_1$; hence, $\Lambda_1(P) = rp_1$ and $\Lambda_2(P) = (n - r) + r(1 - p_1)$. Hence, we obtain that $p_1 = \frac{rp_1 - ((n - r) + r(1 - p_1)) + 1}{2}$, from which we derive:

$$p_1 = \frac{1}{2} + \frac{n - r}{2(r - 1)};$$

$$p_2 = \frac{1}{2} - \frac{n - r}{2(r - 1)}.$$

Clearly, $\Lambda_1(P) = rp_1$ and $\Lambda_2(P) = n - \Lambda_1(P)$. Denote $\alpha = \frac{n}{2}$ and $\beta = \frac{n - r}{2(r - 1)}$, to derive that $\Lambda_1(P) = \alpha + \beta$ and $\Lambda_2(P) = \alpha - \beta$. Then, the average probabilities on links 1 and 2 are $\widetilde{p}_1 = \frac{\Lambda_1(P)}{r} = \frac{\alpha + \beta}{r}$ and $\widetilde{p}_2 = \frac{\Lambda_2(P)}{n} = \frac{\alpha - \beta}{n}$, respectively.

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On one hand, Lemma 2.3 and Proposition 2.2 imply that

$$
PSC_{\lambda}(n, m, P) = BF\left(\left\langle p_1, \ldots, p_r\right\rangle, \lambda^d\right) + BF\left(\left\langle p_{r+1}, \ldots, p_{n-r}\right\rangle, \lambda^d\right)
$$

On the other hand, Lemma 3.2 and Proposition 2.2 imply that

$$
PSC_{\lambda}(n, m, F) = BF\left(\left\langle f_1, \ldots, f_1\right\rangle, \lambda^d\right) + BF\left(\left\langle f_2, \ldots, f_n\right\rangle, \lambda^d\right)
$$

So, clearly,

$$
PSC_{\lambda}(n, m, F) - PSC_{\lambda}(n, m, P) = \sum_{t \in [d]} S(d, t) \cdot \Delta(t),
$$

where for each integer $t \in [d]$,

$$
\Delta(t) = 2\alpha^t \cdot \frac{n^t}{n^t} - \left(\alpha + \beta\right)^t \cdot \frac{r^t}{r^t} + \left(\alpha - \beta\right)^t \cdot \frac{n^t}{n^t}.
$$

We prove:

**Lemma 4.2** For each integer $t \geq 1$, $\Delta(t) \geq 0$.

**Proof:** By induction on $t$. For the basis case where $t = 1$, the claim holds since $2\alpha - ((\alpha + \beta) + (\alpha - \beta)) = 0$. Assume inductively that the claim holds for $(t - 1)$, where $t \geq 2$. For the induction step, we will prove the claim for $t$. 

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Note first that by the definition of $\alpha$ and $\beta$ and since $r \leq n$,
\[
(\alpha + \beta) \cdot \frac{r - (t - 1)}{r} = \frac{(n-1)r}{2(r-1)} \cdot \frac{r - (t - 1)}{r}
\]
\[
= \frac{n - 1}{2} \cdot \frac{r - (t - 1)}{r - 1}
\]
\[
\leq \frac{n - 1}{2} \cdot \frac{n - (t - 1)}{n - 1}
\]
\[
\leq \alpha \cdot \frac{n - (t - 1)}{n - 1}.
\]

We now use this fact to derive that
\[
(\alpha + \beta)^t \cdot \frac{r^t}{r^t} + (\alpha - \beta)^t \cdot \frac{n^t}{n^t}
\]
\[
= (\alpha + \beta) \cdot \frac{r - (t - 1)}{r} \cdot (\alpha + \beta)^{t-1} \cdot \frac{r^{t-1}}{r^{t-1}} + (\alpha - \beta) \cdot \frac{n - (t - 1)}{n} \cdot (\alpha - \beta)^{t-1} \cdot \frac{n^{t-1}}{n^{t-1}}
\]
\[
\leq \alpha \cdot \frac{n - (t - 1)}{n} \cdot (\alpha + \beta)^{t-1} \cdot \frac{r^{t-1}}{r^{t-1}} + \alpha \cdot \frac{n - (t - 1)}{n} \cdot (\alpha - \beta)^{t-1} \cdot \frac{n^{t-1}}{n^{t-1}}
\]
\[
\leq \alpha \cdot \frac{n - (t - 1)}{n} \cdot ((\alpha + \beta)^{t-1} \cdot \frac{r^{t-1}}{r^{t-1}} + (\alpha - \beta)^{t-1} \cdot \frac{n^{t-1}}{n^{t-1}})
\]
\[
\leq \alpha \cdot \frac{n - (t - 1)}{n} \cdot 2\alpha^{t-1} \cdot \frac{n^{t-1}}{n^{t-1}}
\]
\[
= 2\alpha^t \cdot \frac{n^t}{n^t},
\]
where we used the induction hypothesis in the last step, as needed. \qed

Lemma 4.2 implies that $PSC_{\lambda^d}(n, m, \mathbf{P}) \leq PSC_{\lambda^d}(n, m, \mathbf{F})$. The proof for the case $u = 0$ that $PSC_{\pi_d(\lambda)}(n, m, \mathbf{P}) \leq PSC_{\pi_d(\lambda)}(n, m, \mathbf{F})$ is now complete.

(b) Assume now that $u > 0$.

Consider the mixed assignment $\mathbf{Q}$ for the instance $(n, 2)$, which assigns $u$ pure users to each link (with probability 1) and $\tilde{n} = n - 2u$ mixed users to each link with probability $\frac{1}{2}$. Clearly, $\mathbf{Q}$ is a Nash equilibrium. Note that the average probability for each link is $\frac{u \cdot 1 + (n - 2u) \cdot \frac{1}{2}}{n} = \frac{1}{2}$, which is precisely the probability with which each user is assigned to a link in the fully mixed Nash equilibrium $\mathbf{F}$. Since Polynomial Social Cost is a sum of binomial functions and the function $\pi_d(\lambda) = \lambda^d$ is convex, Lemma 2.3 implies that $PSC_{\lambda^d}(n, m, \mathbf{Q}) \leq PSC_{\lambda^d}(n, m, \mathbf{F})$. In the rest, we will prove that $PSC_{\lambda^d}(n, m, \mathbf{P}) \leq PSC_{\lambda^d}(n, m, \mathbf{Q})$, and this will complete the proof.

Denote $\tilde{\mathbf{F}}$ the (unique) fully mixed Nash equilibrium associated with the instance $(\tilde{n}, 2)$. On one hand, $PSC_{\lambda^d}(n, 2, \mathbf{Q}) = PSC_{(\lambda + u)^d}(\tilde{n}, 2, \tilde{\mathbf{F}})$; on the other hand,
\[ \text{PSC}_{\lambda}(\tilde{n}, 2, P) = \text{PSC}_{(\lambda + \alpha)}(\tilde{n}, 2, \tilde{P}), \] where \( \tilde{P} \) is a mixed assignment associated with the instance \( \langle \tilde{n}, 2 \rangle \) that assigns \( v \) pure users to link 2 and \( r \) mixed users to both links. Since \((\lambda + \alpha)^d\) is a convex function (as a linear combination of monomials in \( \lambda \) with non-negative coefficients), we are reduced to Case (a). Hence, it follows that \[ \text{PSC}_{(\lambda + \alpha)}(\tilde{n}, 2, \tilde{P}) \leq \text{PSC}_{(\lambda + \alpha)}(\tilde{n}, 2, F), \] Hence, this implies that \( \text{PSC}_{\lambda}(n, 2, P) \leq \text{PSC}_{\lambda}(n, 2, Q) \), as needed. The proof for the case \( u > 0 \) that \( \text{PSC}_{\sigma_\lambda}(n, m, P) \leq \text{PSC}_{\sigma_\lambda}(n, m, F) \) is now complete.

Since we examined all possible cases, the proof is now complete. \( \blacksquare \)

### 4.2 The Monomial and Polynomial Prices of Anarchy

We first prove:

**Theorem 4.3** Consider the case of identical users and two links. Then,

\[ \text{MPoA} = \frac{1}{2} \left( 2^{d-1} + 1 \right). \]

**Proof:** We start with the upper bound. Fix any instance \( \langle n, m \rangle \) with an associated Nash equilibrium \( P \). Note that, by Theorem 4.1,

\[ \text{PSC}_{\lambda^*}(n, m, P) \leq \text{PSC}_{\lambda^*}(n, m, F) \]
\[ = 2 \cdot \text{BF}(\frac{1}{2} n, \lambda^*) \]
\[ = 2 \cdot \sum_{i \in [d]} \left( \frac{1}{2} \right)^i \cdot S(d, t) \cdot n^\frac{1}{2} \]
\[ \leq 2 \cdot \left( \frac{1}{2} \cdot S(d, 1) \cdot n + \frac{1}{4} \cdot \sum_{3 \leq i \leq d} S(d, i) n^\frac{1}{2} \right) \]
\[ = 2 \cdot \left( \frac{1}{2} \cdot S(d, 1) \cdot n + \frac{1}{4} \cdot (n^d - S(d, 1) \cdot n) \right) \]
\[ = 2 \cdot \left( \frac{n}{4} + \frac{n^d}{4} \right). \]

On the other hand, \( \text{POPT}_{\lambda^*}(n, m) \geq 2 \cdot \left( \frac{n}{2} \right)^d \). It follows that

\[ \frac{\text{PSC}_{\lambda^*}(w, m, F)}{\text{POPT}_{\lambda^*}(w, m)} \leq \left( \frac{2^d}{n} \right) \cdot \left( \frac{n}{4} + \frac{n^d}{4} \right) \]
\[ = \frac{2^d}{4} + \frac{1}{2} \left( \frac{2}{n} \right)^{d-1} \]
\[ \leq \frac{1}{2} \left( 2^{d-1} + 1 \right), \]
as needed for the upper bound. We continue with the proof of the lower bound. Fix an instance \((n, 2)\) with \(n = 2\). Then, \(\text{POPT}_{\lambda}(n, m) = 2\), while

\[
\text{PSC}_{\lambda}(w, m, F) = 2 \cdot \sum_{t \in [\lambda]} \left(\frac{1}{\lambda}\right)^t \cdot S(d, t) \cdot 2^t
\]

\[
= 2 \cdot \left(\frac{1}{2} \cdot S(d, 1) \cdot 2 + \frac{1}{4} \cdot S(d, 2) \cdot 2 \cdot 1\right)
\]

\[
= 2 \cdot \left(S(d, 1) + \frac{1}{2} \cdot S(d, 2)\right)
\]

\[
= 2 \cdot \left(1 + \frac{1}{2} \cdot (2^{d-1} - 1)\right)
\]

\[
= 2 \cdot \left(\frac{1}{2} (2^{d-1} - 1)\right).
\]

It follows that \(\text{MPoA} \geq \frac{1}{2} \left(2^{d-1} - 1\right)\), as needed.

By Lemma 3.1, Theorem 4.3 immediately implies:

**Corollary 4.4** Consider the case of identical users and two links. Then,

\[
\text{PPoA} \leq \frac{1}{2} \left(2^d + d - 1\right).
\]

5 **Identical Users**

In this section, we consider the case of identical users. The \(\text{PFMNE} \) Conjecture is considered in Section 5.1. Bounds on the Monomial and Polynomial Prices of Anarchy are presented in Section 5.2.

5.1 **The \(\text{PFMNE} \) Conjecture**

We prove the validity of an approximate version of the \(\text{PFMNE} \) Conjecture.

**Theorem 5.1** Consider the case of identical users. Fix any instance \((n, m)\) with an associated Nash equilibrium \(P\) and fully mixed Nash equilibrium \(F\). Then,

\[
\text{PSC}_{\pi_{d}(\lambda)}(n, m, P) \leq \left(1 + \frac{1}{n}\right)^{d-1} \cdot \text{PSC}_{\pi_{d}(\lambda)}(n, m, F).
\]

**Proof:** We first consider the case of the monomial latency cost function \(\pi_{d}(\lambda) = \lambda^d\). We will later reduce the general case to this case.

Denote \(\alpha = \frac{n}{m}\). Fix any link \(j \in [m]\), and denote \(r_j = |\{i \in [n]: p_{ij} > 0\}|\) assume, without loss of generality, that \(r_j \geq 1\). Since \(\Lambda_j(P) = \sum_{i \in [n]} p_{ij}\), it follows that the average probability
on link $j$ is $\frac{\Lambda_j(P)}{r_j}$. Define $\beta_j = |\Lambda_j(P) - \alpha|$; roughly speaking, $\beta_j$ is the excess expected latency on link $j$ from the fair share $\alpha$. Partition the set of links $[m]$ into

\[
\mathcal{L}_1 = \{ j \in [m] \mid 0 < \Lambda_j(P) \leq \alpha \},
\]

\[
\mathcal{L}_2 = \{ j \in [m] \mid \Lambda_j(P) > \alpha \}.
\]

Clearly, $\Lambda_j(P) = \alpha - \beta_j$ for $j \in \mathcal{L}_1$ and $\alpha + \beta_j$ for $j \in \mathcal{L}_2$. Define now $q_j = \min_{i \in [n]} \{ p_{ij} \mid p_{ij} > 0 \}$; clearly, $q_j \leq \frac{\Lambda_j(P)}{r_j}$.

Define $\beta = \max_{j \in \mathcal{L}_1} \beta_j$. We prove a simple fact.

**Lemma 5.2** For each link $j \in \mathcal{L}_2$, $\beta_j \leq \frac{\alpha - r_j \beta}{r_j - 1}$.

**Proof:** Fix a link $j \in \mathcal{L}_2$ and a user $i_0 \in \{i \in [n] \mid p_{ij} > 0\}$, such that $p_{i_0j} = q_j$. Consider a link $j' \in \mathcal{L}_1$ such that $\beta_{j'} = \beta$. Since $P$ is a Nash equilibrium, $\mathbf{IC}_{i_0}^j(P) \leq \mathbf{IC}_{i_0}^{j'}(P)$. Clearly, $\mathbf{IC}_{i_0}^j = \Lambda_j(P) - q_j + 1 = \alpha + \beta_j - q_j + 1$; also, $\mathbf{IC}_{i_0}^{j'} = \lambda_j(P) - p_{i_0j'} + 1 = \Lambda_j(P) + 1 = \alpha + \beta_j + 1$. It follows that $\alpha + \beta_j - q_j + 1 \leq \alpha - \beta_j + 1$ or $\beta_j + \beta_j' \leq q_j \leq \frac{\Lambda_j(P)}{r_j} = \frac{\alpha + \beta_j}{r_j}$. This implies that $\beta_j \leq \frac{\alpha - r_j \beta}{r_j - 1}$, as needed. \hfill \blacksquare

On one hand, Lemma 2.3 and Proposition 2.2 imply that

\[
P_{\mathbf{SC}_{\lambda^d}}(n, m, P) = \sum_{i \in [n]} \mathbf{BF}(\langle p_{ij}, \ldots, p_{nj} \rangle, \lambda^d)
\]

\[
= \sum_{j \in \mathcal{L}_1} \mathbf{BF}(\langle p_{ij}, \ldots, p_{nj} \rangle, \lambda^d) + \sum_{j \in \mathcal{L}_2} \mathbf{BF}(\langle p_{ij}, \ldots, p_{nj} \rangle, \lambda^d)
\]

\[
\leq \sum_{j \in \mathcal{L}_1} \mathbf{BF} \left( \frac{\alpha - \beta_j}{r_j}, r_j, \lambda^d \right) + \sum_{j \in \mathcal{L}_2} \mathbf{BF} \left( \frac{\alpha + \beta_j}{r_j}, r_j, \lambda^d \right)
\]

\[
= \sum_{i \in [n]} \sum_{j \in \mathcal{L}_1} \left( \frac{\alpha - \beta_j}{r_j} \right)^{t} \cdot S(d, t) \cdot (r_j)^t + \sum_{i \in [n]} \sum_{j \in \mathcal{L}_2} \left( \frac{\alpha + \beta_j}{r_j} \right)^{t} \cdot S(d, t) \cdot (r_j)^t
\]

\[
= \sum_{i \in [n]} S(d, t) \cdot \left( \sum_{j \in \mathcal{L}_1} \left( \frac{\alpha - \beta_j}{r_j} \right)^{t} \cdot (r_j)^t + \sum_{j \in \mathcal{L}_2} \left( \frac{\alpha + \beta_j}{r_j} \right)^{t} \cdot (r_j)^t \right).
\]

On the other hand Lemma 3.2 and Proposition 2.2 imply that

\[
P_{\mathbf{SC}_{\lambda^d}}(n, m, F) = \sum_{i \in [n]} \mathbf{BF}(\langle f_{ij}, \ldots, f_{nj} \rangle, \lambda^d)
\]

\[
= m \cdot \mathbf{BF} \left( \frac{1}{m^t} n, \lambda^d \right)
\]

\[
= \sum_{i \in [n]} S(d, t) \cdot \left( m \cdot \alpha^t \cdot \frac{n^t}{m^t} \right).
\]

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For each integer $t \in [d]$, define the function
\[
\Delta(t) = \left(\frac{n+1}{n}\right)^{t-1} (m \cdot \alpha^t \cdot \frac{n^t}{n^t}) - \left(\sum_{j \in \mathcal{L}_1} \left(\frac{a - \beta_j}{r_j}\right)^t \cdot (r_j)^t + \sum_{j \in \mathcal{L}_2} \left(\frac{a + \beta_j}{r_j}\right)^t \cdot (r_j)^t\right).
\]

We prove:

**Lemma 5.3** For each integer $t \geq 1$, $\Delta(t) \geq 0$.

**Proof:** By induction on $t$. For the basis case, let $t = 1$. Then,
\[
\Delta(1) = m \cdot a - \left(\sum_{j \in \mathcal{L}_1} (a - \beta_j) + \sum_{j \in \mathcal{L}_2} (a + \beta_j)\right) = m \cdot a - \left(\sum_{j \in \mathcal{L}_1} \Lambda_j(P) + \sum_{j \in \mathcal{L}_2} \Lambda_j(P)\right) = m \cdot a - n = 0,
\]
as needed.

Assume inductively that the claim holds for $(t-1)$, for some integer $t \geq 2$. For the induction step, we will prove the claim for $t$. Since $1^t = 0$ for each $j \in [m]$, assume that $r_j \geq 2$ for each $j \in [m]$. Observe that
\[
\sum_{j \in \mathcal{L}_1} \left(\frac{a - \beta_j}{r_j}\right)^t \cdot (r_j)^t + \sum_{j \in \mathcal{L}_2} \left(\frac{a + \beta_j}{r_j}\right)^t \cdot (r_j)^t = \sum_{j \in \mathcal{L}_1} \left(\frac{a - \beta_j}{r_j}\right)^{t-1} \cdot \frac{r_j - (t-1)}{r_j} \left(\frac{a - \beta_j}{r_j}\right)^{t-1} \cdot (r_j)^{t-1}
+ \sum_{j \in \mathcal{L}_2} \left(\frac{a + \beta_j}{r_j}\right)^{t-1} \cdot \frac{r_j - (t-1)}{r_j} \left(\frac{a + \beta_j}{r_j}\right)^{t-1} \cdot (r_j)^{t-1}.
\]
Note that for each link $j \in [m]$, the fraction $\frac{r_j - (t-1)}{r_j}$ is monotonically increasing in $r_j$ (since $t \geq 2$). Since for each $j \in [m]$, $r_j \leq n$, it follows that
\[
\sum_{j \in \mathcal{L}_1} \left(\frac{a - \beta_j}{r_j}\right)^t \cdot (r_j)^t + \sum_{j \in \mathcal{L}_2} \left(\frac{a + \beta_j}{r_j}\right)^t \cdot (r_j)^t \leq \sum_{j \in \mathcal{L}_1} a \cdot \left(\frac{n - (t-1)}{n}\right)^{-1} \left(\frac{a - \beta_j}{r_j}\right)^{t-1} \cdot (r_j)^{t-1}
+ \sum_{j \in \mathcal{L}_2} \left(\frac{a + \beta_j}{r_j}\right)^{t-1} \cdot \frac{r_j - (t-1)}{r_j} \left(\frac{a + \beta_j}{r_j}\right)^{t-1} \cdot (r_j)^{t-1}.
\]

We proceed by case analysis.
1. Assume that for each link \( j \in \mathcal{L}_2 \), \( \beta_j \leq \frac{n - r_j}{m(r_j - 1)} \). The assumption implies that for each link \( j \in \mathcal{L}_2 \), \( \alpha + \beta_j \leq \frac{(n - 1)r_j}{m(r_j - 1)} \). Then,

\[
\sum_{j \in \mathcal{L}_1} \frac{(a - \beta_j)}{r_j} \left( n \right) \frac{t}{r_j} + \sum_{j \in \mathcal{L}_2} \frac{(a + \beta_j)}{r_j} \left( n \right) \frac{t}{r_j} \\
\leq \sum_{j \in \mathcal{L}_1} \frac{a - (t - 1)}{n} \left( \frac{a - \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} + \sum_{j \in \mathcal{L}_2} \frac{n - (t - 1)}{m} \frac{r_j}{m(r_j - 1)} \left( \frac{a + \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} \\
= \sum_{j \in \mathcal{L}_1} \frac{a - (t - 1)}{n} \left( \frac{a - \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} + \sum_{j \in \mathcal{L}_2} \frac{n - (t - 1)}{m} \frac{r_j}{m(r_j - 1)} \left( \frac{a + \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} \\
\leq \sum_{j \in \mathcal{L}_1} \frac{a - (t - 1)}{n} \left( \frac{a - \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} + \sum_{j \in \mathcal{L}_2} \frac{n - (t - 1)}{m} \frac{r_j}{m(r_j - 1)} \left( \frac{a + \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} \\
= \frac{a - (t - 1)}{n} \left( \sum_{j \in \mathcal{L}_1} \frac{a - \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} + \sum_{j \in \mathcal{L}_2} \frac{n - (t - 1)}{m} \frac{r_j}{m(r_j - 1)} \left( \frac{a + \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} \\
\leq \frac{a - (t - 1)}{n} \left( \sum_{j \in \mathcal{L}_1} \frac{a - \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} + \sum_{j \in \mathcal{L}_2} \frac{n - (t - 1)}{m} \frac{r_j}{m(r_j - 1)} \left( \frac{a + \beta_j}{r_j} \right) \left( n \right) \frac{t}{r_j} \\
< \left( \frac{n + 1}{n} \right) \frac{t}{n} \left( m \right) \frac{t}{n} \frac{t}{n},
\]

where the induction hypothesis was used for the last inequality. This implies that \( \Delta(t) \geq 0 \), as needed.

2. Assume now that there exists a link \( j \in \mathcal{L}_2 \) such that \( \beta_j > \frac{n - r_j}{m(r_j - 1)} \).

We first prove that \( \beta \leq \frac{1}{m} \). Assume, by way of contradiction, that \( \beta > \frac{1}{m} \). Then, by Lemma 5.2, \( \beta_j \leq \frac{\alpha - r_j}{r_j - 1} \leq \frac{n - r_j}{m} \frac{r_j}{m(r_j - 1)} < \beta_j \), a contradiction. It follows that for each link \( j \in \mathcal{L}_2 \),

\[
(a + \beta_j) \cdot \frac{r_j - (t - 1)}{r_j} \leq (a + \beta) \cdot \frac{n - (t - 1)}{n} \\
\leq \left( a + \frac{1}{m} \right) \cdot \frac{n - (t - 1)}{n} \\
= \frac{n + 1}{n} \cdot \alpha \cdot \frac{n - (t - 1)}{n}.
\]
Thus,

\[
\sum_{j \in L_1} \left( \frac{\alpha - \beta_j}{r_j} \right)^t (r_j)^k + \sum_{j \in L_2} \left( \frac{\alpha + \beta_j}{r_j} \right)^t (r_j)^k \leq \sum_{j \in L_1} \frac{n - (t - 1)}{n} \left( \frac{\alpha - \beta_j}{r_j} \right)^{t-1} (r_j)^{k-1} + \sum_{j \in L_2} \frac{n + 1}{n} \frac{n - (t - 1)}{n} \left( \frac{\alpha + \beta_j}{r_j} \right)^{t-1} (r_j)^{k-1} \\
\leq \frac{n + 1}{n} \frac{n - (t - 1)}{n} \left( \sum_{j \in L_1} \left( \frac{\alpha - \beta_j}{r_j} \right)^{t-1} (r_j)^{k-1} + \sum_{j \in L_2} \left( \frac{\alpha + \beta_j}{r_j} \right)^{t-1} (r_j)^{k-1} \right) \\
\leq \frac{n + 1}{n} \frac{n - (t - 1)}{n} \left( \frac{n + 1}{n} \right)^{t-2} \left( m \cdot \alpha^{t-1} \cdot \frac{n^t}{n^{t-1}} \right) \\
= \left( \frac{n + 1}{n} \right)^{t-1} \left( \frac{n}{m} \right),
\]

where the induction hypothesis was used for the last inequality. This implies that \( \Delta(t) \geq 0 \), as needed.

\[\square\]

Lemma 5.3 implies that \( \text{PSC}_{\lambda, \rho}(n, m, P) \leq \left( 1 + \frac{1}{n} \right)^d \cdot \text{PSC}_{\lambda, \rho}(n, m, F) \). Hence,

\[
\text{PSC}_{\pi, \rho}(n, m, P) = \sum_{0 \leq t \leq d} \alpha_t \cdot \text{PSC}_{\lambda, \rho}(n, m, P) \\
\leq \sum_{0 \leq t \leq d} \alpha_t \cdot \left( 1 + \frac{1}{n} \right)^{t-1} \cdot \text{PSC}_{\lambda, \rho}(n, m, F) \\
\leq \left( 1 + \frac{1}{n} \right)^{d-1} \cdot \sum_{0 \leq t \leq d} \alpha_t \cdot \text{PSC}_{\lambda, \rho}(n, m, F) \\
\leq \left( 1 + \frac{1}{n} \right)^{d-1} \cdot \text{PSC}_{\pi, \rho}(n, m, F),
\]

as needed.

\[\square\]

We remark that the proof of Theorem 5.1 follows the proof of Theorem 4.1. However, it is more complicated in dealing with an arbitrary number of links. We believe that the resulting additional factor \( \left( 1 + \frac{1}{n} \right)^{d-1} \) is not necessary.

### 5.2 The Monomial and Polynomial Prices of Anarchy

We prove:
**Theorem 5.4** Consider the case of identical users. Then,

\[ MPoA \leq \left(1 + \frac{1}{n}\right)^{d-1} \cdot B_d. \]

**Proof:** Fix any instance \((n, m)\) with an associated fully mixed Nash equilibrium \(F\). By Proposition 2.2,

\[
PSC_{\lambda^d}(n, m, F) = \sum_{j \in [m]} BF(\langle f_{i_j}, \ldots, f_{i_j} \rangle, \lambda^d) = m \cdot BF(\langle f_{i_j}, \ldots, f_{i_j} \rangle, \lambda^d) = m \cdot \sum_{t \in [d]} \left(\frac{1}{m}\right)^t \cdot S(d, t) \cdot n^t \leq m \cdot \sum_{t \in [d]} \left(\frac{1}{m}\right)^t \cdot S(d, t) \cdot n^t.
\]

We now proceed by case analysis.

1. Assume first that \(n \geq m\). Recall that in this case, \(\text{POPT}_{\lambda^d}(w, m) \geq m \cdot \left(\frac{n}{m}\right)^d\). Hence,

\[
\frac{PSC_{\lambda^d}(n, m, F)}{\text{POPT}_{\lambda^d}(n, m)} \leq \frac{1}{m} \cdot \left(\frac{m}{n}\right)^d \cdot m \cdot \sum_{t \in [d]} \left(\frac{1}{m}\right)^t \cdot S(d, t) \cdot n^t
= \sum_{t \in [d]} \left(\frac{m}{n}\right)^{d-t} \cdot S(d, t)
\leq \sum_{t \in [d]} S(d, t)
= B_d.
\]

2. Assume now that \(n < m\). Recall that, in this case, \(\text{POPT}_{\lambda^d}(n, m) \geq n\). Hence,

\[
\frac{PSC_{\lambda^d}(n, n, F)}{\text{POPT}_{\lambda^d}(n, m)} \leq \frac{1}{n} \cdot m \cdot \sum_{t \in [d]} \left(\frac{1}{m}\right)^t \cdot S(d, t) \cdot n^t
= \sum_{t \in [d]} \left(\frac{n}{m}\right)^{t-1} \cdot S(d, t)
\leq \sum_{t \in [d]} S(d, t)
= B_d.
\]
So, in all cases, \( \frac{\text{PSC}_{\lambda}(n, m, F)}{\text{POPT}_{\lambda}(n, m)} \leq B_d \). Theorem 5.1 implies now the claim. \( \blacksquare \)

Note that the upper bound on Monomial Price of Anarchy established in Theorem 5.4 approaches \( B_d \) as \( n \) approaches infinity. By Lemma 3.1, Theorem 5.4 immediately implies:

**Corollary 5.5** Consider the case of identical users. Then,

\[
\text{PPoA} \leq \sum_{i \in [d]} \left(1 + \frac{1}{n}\right)^{t-1} \cdot B_i.
\]

### 6 Pure Nash Equilibria

In this section, we consider pure Nash equilibria.

We first recall some technical definition from [13]. For a given instance \((w, m)\), Call a user \( i \in [n] \) **bursty** [13, Section 3] if \( w_i > \frac{W_m}{m} \). Intuitively, the traffic of a bursty user exceeds the fair share of traffic for a link. Say that an instance \((w, m)\) is **bursty** if some user \( i \in [n] \) is bursty; else, \((w, m)\) is **non-bursty**. We first prove a simple property of bursty users.

**Lemma 6.1** A bursty user is solo in an optimal assignment.

**Proof:** Fix an instance \((w, m)\) with bursty user \( i \in [n] \). Consider an optimal assignment \( Q = \langle q_1, \ldots, q_n \rangle \). Note that \( \lambda_{q_i}(Q) \geq w_i \). Since \( i \) is bursty, it follows that \( \lambda_{q_i}(Q) > \frac{W_m}{m} \). Since \( \sum_{j \in [m]} \lambda_j(Q) = W \), there is some other link \( j \in [m] \) with \( j \neq i \) such that \( \lambda_j(Q) < \frac{W_m}{m} \). Assume now, by way of contradiction, that some user \( k \neq i \) is assigned to link \( q_i \). Modify \( Q \) to obtain \( Q' \) by switching user \( k \) to link \( j \). Then,

\[
\text{PSC}_{\lambda}(w, m, Q') - \text{PSC}_{\lambda}(w, m, Q) = \sum_{1 \leq t \leq d} \left( \left(\lambda_{q_i}(Q')\right)^t + \left(\lambda_j(Q')\right)^t - \left(\lambda_{q_i}(Q)\right)^t - \left(\lambda_j(Q)\right)^t \right)
\]

\[
= \sum_{1 \leq t \leq d} \left( \left( w_i \right)^t + \left( w_k \right)^t + \left( w_j \right)^t + \left( \lambda_j(Q) + w_k \right)^t - \left( w_i + w_k \right)^t - \left( \lambda_j(Q) \right)^t \right).
\]

Since \( \lambda_j(Q) < w_i \) and the function \( f(x) = (x+a)^t - x^t \) (with \( a \geq 0 \)) is monotonically increasing in \( x \), it follows that

\[
\text{PSC}_{\lambda}(w, m, Q') - \text{PSC}_{\lambda}(w, m, Q) < \sum_{0 \leq t \leq d} \left( \left( w_i \right)^t + \left( w_i + w_k \right)^t - \left( w_i + w_k \right)^t - \left( w_i \right)^t \right)
\]

\[
= 0.
\]

Since \( Q \) is optimum, \( \text{PSC}_{\lambda}(w, m, Q') \geq \text{PSC}_{\lambda}(w, m, Q) \). A contradiction. \( \blacksquare \)

We remark that the proof of Lemma 6.1 implicitly assumes a monomial latency cost function. The proof carries over to the general case with only a few modifications.

The following two simple properties relating bursty users and pure Nash equilibria were shown in [13, Section 3]; they carry over to our model since their sets of Nash equilibria coincide.
Lemma 6.2 A bursty user is solo in a pure Nash equilibrium.

Lemma 6.3 Consider a pure Nash equilibrium $\mathbf{P}$ for a non-bursty instance $(\mathbf{w}, m)$. Then, for each link $j \in [m]$, $\lambda_j(\mathbf{P}) \leq 2 \min_{\ell \in [m]} \lambda_\ell(\mathbf{L})$.

We are now ready to prove:

Theorem 6.4 For pure Nash equilibria,

$$\text{MPoA} = \frac{(2^d - 1)^d}{(d-1)(2^d-2)^{d-1}} \left( \frac{d-1}{d} \right)^d.$$ 

In our proof, we will make use of the following notations. Consider an instance $(\mathbf{w}, m)$ with an associated pure assignment $\mathbf{L}$. Fix a set of links $\mathcal{L}$, inducing a set of users $\mathcal{U}$ that are assigned by $\mathbf{L}$ to links in $\mathcal{L}$. Then, $\mathbf{w} \setminus \mathcal{U}$ and $m \setminus \mathcal{L}$ denote the traffic vector and the number of links resulting from $\mathbf{w}$ and $m$, respectively, by respective eliminations of all entries corresponding to users in $\mathcal{U}$ and links in $\mathcal{L}$. Also, $\mathbf{L} \setminus (\mathcal{U}, \mathcal{L})$ denotes the assignment induced by these eliminations.

We are now ready for the proof.

Proof: We first prove the upper bound. Consider any arbitrary instance $(\mathbf{w}, m)$ with associated pure Nash equilibrium $\mathbf{L} = (\ell_1, \ldots, \ell_n)$ and optimal assignment $\mathbf{Q} = (q_1, \ldots, q_n)$. Denote $\lambda(\mathbf{L}) = \min_{\ell \in [m]} \lambda_\ell(\mathbf{L})$. We distinguish between two cases:

1. The instance $(\mathbf{w}, m)$ is non-bursty:

Recall that in this case, by Lemma 6.3, for each link $j \in [m]$, $\lambda_j(\mathbf{L}) \leq 2 \lambda(\mathbf{L})$. So, transform the set of loads $\{\lambda_\ell(\mathbf{L}) | \ell \in [m]\}$ into a new set of loads $\{\hat{\lambda}_\ell | \ell \in [m]\}$ as the output of the following repetitive procedure:

```plaintext
for each link $\ell \in [m]$ do
  $\hat{\lambda}_\ell = \lambda(\mathbf{L})$;
while there are distinct links $j_1, j_2 \in [m]$ with $\lambda(\mathbf{L}) < \hat{\lambda}_{j_1} \leq \hat{\lambda}_{j_2} < 2 \lambda(\mathbf{L})$ do
  $\hat{\lambda}_{j_1} = \hat{\lambda}_{j_1} - \min\{\hat{\lambda}_{j_1} - \lambda(\mathbf{L}), 2 \lambda(\mathbf{L}) - \hat{\lambda}_{j_2}\}$;
  $\hat{\lambda}_{j_2} = \hat{\lambda}_{j_2} + \min\{\hat{\lambda}_{j_1} - \lambda(\mathbf{L}), 2 \lambda(\mathbf{L}) - \hat{\lambda}_{j_2}\}$.
```

Intuitively, our transformation procedure chooses at each step two intermediate latencies $\hat{\lambda}_{j_1}$ and $\hat{\lambda}_{j_2}$ (that is, two latencies that are not yet pushed either to the upper or to the lower end of the interval of link loads). It transfers the (strictly) positive quantity $\min\{\hat{\lambda}_{j_1} - \lambda(\mathbf{L}), 2 \lambda(\mathbf{L}) - \hat{\lambda}_{j_2}\}$ from the small latency $\hat{\lambda}_{j_1}$ to the large latency $\hat{\lambda}_{j_2}$. Clearly,
each step of the procedure either pushes the small latency $\hat{\lambda}_{j_1}$ to the lower end $\lambda(L)$ of the interval of link latencies, or pushes the large load $\hat{\lambda}_{j_2}$ to the upper end $2\lambda(L)$ of the interval of link latencies (or both). So, clearly, when the procedure terminates, there is at most one intermediate latency. Hence, by reordering links, we obtain that for some integer $k \in [m-1] \cup \{0\}$, for each link $j \in [m]$,

$$
\hat{\lambda}_j = \begin{cases} 
2\lambda(L) & \text{if } j \in [k] \\
(1+x)\lambda(L) & \text{if } j = k + 1 \\
\lambda(L) & \text{if } j \in [m] \setminus [k+1],
\end{cases}
$$

where $0 \leq x < 1$. Intuitively, $k$ is the number of overloaded links.

Note that this transformation procedure maps a set of latencies to a new set of latencies, without explicitly mapping an instance to a new instance. However, for the sake of our analysis, we will also consider that the procedure maps an instance $(w, m)$ and a Nash equilibrium $L$ to a new instance $(\hat{w}, m)$ and a new Nash equilibrium $\hat{L}$. Note also that this transformation preserves (at each of its steps) the sum of latencies. Hence, it also preserves the total latencies, so that $W = \hat{W}$.

For any individual step of our repetitive procedure, Proposition 2.1 implies that

$$
PSC_{\lambda,d}(\hat{w}, m, \hat{L}) - PSC_{\lambda,d}(w, m, L) = 
\left(\left(\hat{\lambda}_{j_1} - \min\{\hat{\lambda}_{j_1}, \lambda(L), 2\lambda(L) - \hat{\lambda}_{j_2}\}\right)^d + \left(\hat{\lambda}_{j_2} + \min\{\hat{\lambda}_{j_1} - \lambda(L), 2\lambda(L) - \hat{\lambda}_{j_2}\}\right)^d\right)
- \left(\hat{\lambda}_{i_1}^d - \hat{\lambda}_{i_2}^d\right) > 0.
$$

Hence, it follows that,

$$
PSC_{\lambda,d}(w, m, L) \leq PSC_{\lambda,d}(\hat{w}, m, \hat{L}) = \sum_{j \in [m]} \left(\lambda_j(\hat{L})\right)^d = k(2\lambda(L))^d + ((1+x)\lambda(L))^d + (m - k - 1)\lambda(L)^d = \left(m + (2^d - 1)k - 1 + (1+x)^d\right)\lambda(L)^d.
$$
On the other hand,

\[
\text{POPT}_{\lambda,d}(\mathbf{w}, m) \geq \frac{(W/m)^d}{m} = \frac{\hat{W}d}{m^{d-1}} = \frac{\left(\sum_{j \in [m]} \lambda_j(\hat{L})\right)^d}{m^{d-1}} = \frac{(m + k + x)^d\lambda(L)^d}{m^{d-1}}.
\]

It follows that

\[
\text{PPOA} \leq \frac{(m + (2^d - 1)k - 1 + (1 + x)^d)m^{d-1}}{(m + k + x)^d}.
\]

Define the real function

\[
f(k) = \frac{(m + (2^d - 1)k - 1 + (1 + x)^d)m^{d-1}}{(m + k + x)^d}
\]

of a real variable \(k\). (The quantity \(x\) is taken as a parameter, while \(m\) is a fixed constant). Clearly, \(\text{MPOA} \leq \sup_k f(k)\). So, we will determine \(\sup_k f(k)\).

To maximize the function \(f(k)\), observe that the first and second derivatives of \(f(k)\) are

\[
f'(k) = \frac{(2^d - 1)m^{d-1}}{(m + k + x)^d} - \frac{(m + (2^d - 1)k - 1 + (1 + x)^d)m^{d-1}d}{(m + k + x)^{d+1}}
\]

and

\[
f''(k) = \frac{m^{d-1}d((2^d - 1)(d-1)k - 2(2^d - 1)(m + x) + (m - 1 + (1 + x)^d)(d + 1))}{(m + k + x)^{d+2}},
\]

respectively. The only root of \(f'(k)\) is

\[
k_0 = \frac{(2^d - 1)(m + x) + d(-m + 1 - (1 + x)^d)}{(2^d - 1)(d - 1)}.
\]

For \(k = k_0\), the second derivative evaluates to

\[
f''(k_0) = \frac{m^{d-1}d(-m(2^d - 2) - (2^d - 1)x + (1 + x)^d - 1)}{(m + k_0 + x)^{d+2}}.
\]

Since \(-(2^d - 1)x + (1 + x)^d \leq 2^d\) holds for all \(x \in [0, 1]\), it follows that \(f''(k_0) < 0\). Thus, \(k_0\) is a local maximum of the function \(f(k)\). Since \(f(k)\) is a continuous function with a single extreme point that is a local maximum, it follows that

\[
f(k) \leq f(k_0) = \frac{(2^d - 1)^d}{d - 1} \left(\frac{d - 1}{d}\right)^d \cdot \frac{m^{d-1}}{m(2^d - 2) + x(2^d - 1) - (1 + x)^d + 1}^{d-1}.
\]
Note that the minimum value of the function \( h(x) = x(2^d - 1) - (1 + x)^d + 1 \) for \( x \in [0, 1] \) is \( h(0) = h(1) = 0 \). Thus

\[
    f(k) \leq \frac{(2^d - 1)^d}{d - 1} \cdot \left( \frac{d - 1}{d} \right)^d \cdot \frac{m^{d-1}}{(m(2^d - 2))^{d-1}}
\]

as needed.

2. The instance \( \langle w, m \rangle \) is bursty:

Denote \( \mathcal{U} \) the (non-empty) set of bursty users. Recall that, by Lemmas 6.1 and 6.2, \( \mathcal{U} \) induces sets of solo links \( \mathcal{L}_L \) and \( \mathcal{L}_Q \) for the Nash equilibrium \( L \) and the optimal assignment \( Q \), respectively, so that \( |\mathcal{L}_L| = |\mathcal{U}| \) and \( |\mathcal{L}_Q| = |\mathcal{U}| \). Since links are identical, we assume that \( \mathcal{L}_L = \mathcal{L}_Q = \mathcal{L} \), with \( |\mathcal{L}| \geq 1 \). So,

\[
    \text{PSC}_{\lambda}(w, m, L) = \sum_{j \in \mathcal{L}} (\lambda_j(L))^d + \text{PSC}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}, L \setminus (\mathcal{U}, \mathcal{L}))
\]

and

\[
    \text{POPT}_{\lambda}(w, m) = \text{PSC}_{\lambda}(w, m, Q)
    = \sum_{j \in \mathcal{L}} (\lambda_j(L))^d + \text{PSC}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}, Q \setminus (\mathcal{U}, \mathcal{L}))
    = \sum_{i \in \mathcal{U}} w_i^d + \text{PSC}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}, Q \setminus (\mathcal{U}, \mathcal{L})).
\]

Note first that the assignment \( L \setminus (\mathcal{U}, \mathcal{L}) \) is a Nash equilibrium for the instance \( \langle w \setminus \mathcal{U}, [m] \setminus \mathcal{L} \rangle \). Moreover, since \( Q \) is an optimal assignment for the instance \( \langle w, m \rangle \), it follows that \( Q \setminus (\mathcal{U}, \mathcal{L}) \) is an optimal assignment for the instance \( \langle w \setminus \mathcal{U}, [m] \setminus \mathcal{L} \rangle \), so that

\[
    \text{PSC}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}, Q \setminus (\mathcal{U}, \mathcal{L})) = \text{POPT}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}).
\]

Thus,

\[
    \text{POPT}_{\lambda}(w, m) = \sum_{i \in \mathcal{U}} w_i^d + \text{POPT}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}).
\]

It follows that

\[
    \frac{\text{PSC}_{\lambda}(w, m, L)}{\text{POPT}_{\lambda}(w, m)} = \frac{\sum_{i \in \mathcal{U}} w_i^d + \text{PSC}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}, L \setminus (\mathcal{U}, \mathcal{L}))}{\sum_{i \in \mathcal{U}} w_i^d + \text{POPT}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L})}
\]

\[
\leq \frac{\text{PSC}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L}, L \setminus (\mathcal{U}, \mathcal{L}))}{\text{POPT}_{\lambda}(w \setminus \mathcal{U}, [m] \setminus \mathcal{L})}.
\]
So consider the instance \( \langle w \setminus U, [m] \setminus L \rangle \) and the associated pure Nash equilibrium \( L \setminus (U, L) \). There are two possibilities depending on whether the smaller instance \( \langle w \setminus U, [m] \setminus L \rangle \) is bursty or not.

- Assume first that the instance \( \langle w \setminus U, [m] \setminus L \rangle \) is non-bursty. Then, we are reduced to the previous case of non-bursty instances, and the upper bound follows.
- Assume now that the smaller instance \( \langle w \setminus U, [m] \setminus L \rangle \) is bursty. We repeatedly identify the set of bursty users for the smaller instance, and we reduce this smaller instance to an even smaller instance that may be bursty or non-bursty. This procedure eventually yields a non-bursty instance (even the trivial one with one user), and the claim for the original bursty instance follows inductively.

The proof of the upper bound is now complete.

We continue to prove the lower bound. Construct an instance \( \langle w, m \rangle \) as follows. There are \( m = (2^d - 1)(d - 1) \) links. There are \( 2(2^d - d - 1) \) heavy users with traffic 1; there are \( m \cdot (m - (2^d - d - 1)) \) light users with traffic \( \frac{1}{m} \). Consider now the following assignments:

- In the pure assignment \( L \), heavy users are evenly distributed to \( 2^d - d - 1 \) links; light users are evenly distributed to the remaining \( m - (2^d - d - 1) \) links. Clearly, \( L \) is a Nash equilibrium with
  
  \[
  \text{PSC}_{\lambda,d}(w, m, L) = 2^d \cdot (2^d - d - 1) + 1^d \cdot ((2^d - 1)(d - 1) - (2^d - d - 1)) 
  = (2^d - 1)(2^d - 2). 
  \]

- In the pure assignment \( Q \), each (of \( 2(2^d - d - 1) \)) heavy user is assigned solo to each of \( 2(2^d - d - 1) \) links; \( m(m - 2(2^d - d - 1)) \) light users are evenly assigned to the remaining \( m - 2(2^d - d - 1) \) links, while the remaining \( m(2^d - d - 1) \) light users are evenly assigned to all \( m \) links. It is easy to see that the latency on each link induced by \( Q \) is \( 1 + \frac{2^d - d - 1}{m} = \frac{m + 2^d - d - 1}{m} \). Thus,
  
  \[
  \text{PSC}_{\lambda,d}(w, m, Q) = m \cdot \left( \frac{m + 2^d - d - 1}{m} \right)^d 
  = \frac{(d - 1)(2^d - 2)^d}{(2^d - 1)^{d-1}} \cdot \left( \frac{d}{d - 1} \right)^d. 
  \]

Thus,

\[
\text{MPoA} \geq \frac{\text{PSC}_{\lambda,d}(w, m, L)}{\text{PSC}_{\lambda,d}(w, m, Q)} 
= \frac{(2^d - 1)^d}{(d - 1)(2^d - 2)^{d-1}} \cdot \left( \frac{d}{d - 1} \right)^d,
\]

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as needed.

By Lemma 3.1, Theorem 6.4 immediately implies:

**Corollary 6.5** *For pure Nash equilibria.*

\[
\text{PPoA} \leq \sum_{2 \leq i \leq d} \frac{(2^i - 1)^i}{(l-1)(2^i - 2)^i-1} \left( \frac{t-1}{t} \right)^i.
\]

7 **Epilogue**

We introduced and analyzed an interesting variant of the well studied KP model [12] for selfish routing that reflects some influence from the much older Wardrop model [21]. Our analysis highlights some interesting connections to classical combinatorial numbers such as the Stirling numbers of the second kind [20] and the Bell numbers [1]. In particular, we formulated and proved exact and approximate versions of the PFMNE Conjecture. In turn, these versions were instrumental for proving (sometimes tight) bounds on Monomial Price of Anarchy; these immediately implied upper bounds on Polynomial Price of Anarchy.

Several interesting problems remain open. On the most concrete level, we are missing a proof of the PFMNE Conjecture for the general case of an arbitrary number of links. We believe that the proof will need some new insights. Also, we do not yet know any lower bounds on Polynomial Price of Anarchy. Are our upper bounds tight? We are also missing *general* bounds on Monomial and Polynomial Prices of Anarchy (ones that hold for arbitrary users, for an arbitrary number of links and for all (mixed) Nash equilibria). Proving such bounds remains a very challenging open problem.
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