The unsatisfiability threshold conjecture: techniques behind upper bound improvements

Lefteris M. Kirousis, Yannis C. Stamatiou and Michele Zito

2004
The Satisfiability Threshold Conjecture: 
Techniques Behind Upper Bound Improvements

Lefteris M. Kirousis
Yannis C. Stamatiou
Michele Zito

1 INTRODUCTION

One of the most challenging problems in probability and complexity theory is 
to establish and determine the satisfiability threshold, or phase transition, for 
random $k$-SAT instances: Boolean formulas consisting of clauses with exactly $k$ 
literals. As the previous part of the volume has explored, empirical observations 
suggest that there exists a critical ratio of the number of clauses to the number 
of variables, such that almost all randomly generated formulas with a higher 
 ratio are unsatisfiable while almost all randomly generated formulas with a lower 
ratio are satisfiable. The statement that such a crossover point really exists is 
called the satisfiability threshold conjecture. Experiments hint at such a direction, 
but as far as theoretical work is concerned, progress has been difficult. In an 
important advance, Friedgut [23] showed that the phase transition is a sharp one, 
though without proving that it takes place at a “fixed” ratio for large formulas. 
Otherwise, rigorous proofs have focused on providing successively better upper 
and lower bounds for the value of the (conjectured) threshold. In this chapter, our
2 The Satisfiability Threshold Conjecture

goal is to review the series of improvements of upper bounds for 3-SAT and the
techniques leading to these. We give only a passing reference to the improvements
of the lower bounds as they rely on significantly different techniques, some of
which are discussed in later chapters.

Let \( \phi \) be a random \( k \)-SAT formula constructed by selecting, uniformly and
with replacement, \( m \) clauses from the set of all possible clauses with \( k \) literals
(no variable repetitions allowed within a clause) over \( n \) variables. It has been
experimentally observed that as the numbers \( n, m \) of variables and clauses tend
to infinity while the ratio or clause density \( m/n \) is fixed to a constant \( \alpha \), the
property of satisfiability exhibits a phase transition. For the case of 3-SAT, when
\( \alpha \) is greater than a number that has been experimentally determined to be ap-
proximately 4.27, then almost all random 3-SAT formulas are unsatisfiable, that
is, the fraction of unsatisfiable formulas tends to 1. The opposite is true when
\( \alpha < 4.27 \). Analogous phenomena have been observed for \( k \)-SAT with \( k > 3 \),
and the experimentally determined threshold point increases with \( k \). The experi-
ments that led to these conclusions were initiated by the work of Cheeseman et
al. [10]. For detailed numerical results see Crawford and Auton [14] and Mitchell
et al. [44]. For \( k = 2 \), it has been “rigorously” established independently by
Chvátal and Reed [11], Goerdt [28, 27], and Fernandez de la Vega [20] that
a transition from almost certain satisfiability to almost certain unsatisfiability
takes place at a clause-to-variable ratio equal to 1.

For \( k \geq 3 \), finding the exact value of the threshold point where this transition
occurs—or even proving that such a threshold exists—is still an open problem.
The following is known. Friedgut [23] has shown that for \( k \)-SAT the transition is
sharp, so that in the large \( n \) limit, the probability of satisfiability changes from
arbitrarily close to 1 to arbitrarily close to 0, as the density \( \alpha \) moves along arbi-
trarily short intervals. However, it is not known whether these intervals converge
to a fixed point. Also, Istrate [30] has shown that the transition is first order: as
\( \alpha \) moves along these intervals of asymptotically zero length, the value of a certain
combinatorial parameter of the random formula jumps from zero to a nonzero
multiple of \( n \). Such parameters are called order parameters in statistical physics.
The specific one used in 3-SAT is the size of the formula’s spine, defined as the
set of all literals \( l \) for which a subformula \( \psi \subseteq \phi \) can be found, so that \( l \) is false
in every truth assignment satisfying \( \psi \). Furthermore, recent theoretical work in
statistical physics [43] has supplied additional and almost conclusive evidence—
though not a formal proof in the mathematical sense—for the existence of the
threshold point. Some of this has been discussed in Braunstein [6].

Apart from the results above, much effort has been put into rigorously estab-
lishing upper and lower bounds for the region where the \( k \)-SAT transition occurs.
These efforts have resulted in interesting and novel probabilistic techniques. In
this chapter we will mainly concentrate on presenting the upper-bound results
and the techniques that lead to them (see also the review by Dubois [15]).
2 GENERATING RANDOM 3-SAT FORMULAS

Let $\Omega$ denote the set of all $2^3(\binom{n}{3})$ possible 3-SAT clauses. A random 3-SAT formula $\phi$ on $m = \alpha n$ clauses can be formed using one of the following frequently employed probability models:

1. Model $\mathcal{G}_{m,m}$: select the $m$ clauses of $\phi$ by drawing them uniformly at random, independently of one another and with replacement, from $\Omega$;
2. Model $\mathcal{G}_m$: as above, but with no replacement;
3. Model $\mathcal{G}_p$: place each clause of $\Omega$ in $\phi$ independently of the others and with probability $p$; and
4. Model $\mathcal{G}_{3,\alpha n}$: fill each of the $3\alpha n$ possible literal positions ($\alpha n$ clauses each having 3 literals) with literals chosen uniformly at random, independently and with replacement, from the set of $2n$ possible literals over the $n$ variables. Note that this model allows the formation of clauses containing variable repetitions.

All of these models are variations on the fixed clause length model introduced by Franco and Paull [22]. That model was an adaptation of the classical model for random graphs introduced by Erdős and Rényi in a series of seminal papers published in the 1960s (see the book of Bollobás [5] for the historical development of the field of random graphs).

The fixed clause length model is in sharp contrast to the variable clause length model introduced by Goldberg et al. [29] in order to study the average time complexity of satisfiability algorithms. In the variable clause length model, each of a “fixed” number of clauses is formed by placing every possible literal in the clause with some probability, and independently of the others. This model has the disadvantage of inducing, on the set of all Boolean formulas with given $n$, a probability distribution that favors easy instances. The fixed clause length model does not have this feature, since it allows the manipulation of the instance hardness by means of the clause density parameter (clause-to-variable ratio) $\alpha$.

Each of these models has its own distinct advantages and disadvantages. The model $\mathcal{G}_{m,m}$ usually leads to tighter results than $\mathcal{G}_p$. On the other hand, the latter has the important property of independence for events involving “non-intersecting” sets of clauses, events which may be dependent in $\mathcal{G}_{m,m}$. Finally, as we will see later, $\mathcal{G}_{3,\alpha n}$ enables one to study as well as manipulate individual literal appearances in a formula. This fact leads to a finer description of the formula than the detail that the other models can achieve. As we discuss in section 6, this may lead to better upper bound values as one usually applies the techniques we will examine on a more limited and well-defined set of formulas. However, it can be shown that if a threshold exists in any one of the above models, it exists in all of them and its value is equal in all of them, even though the bounds obtained by a given method may differ from model to model.

In the sections where we examine the rigorous techniques that have been used in order to bound the satisfiability threshold from above, we will see examples...
of the advantages and disadvantages mentioned and how they are exploited or circumvented, respectively. Unless stated otherwise, we will assume throughout this chapter that we work with the model $G_{m,m}$.

3 THE SUCCESSIVE THRESHOLD APPROXIMATIONS

For the purposes of this chapter, we will accept the satisfiability threshold conjecture, and denote the $k$-SAT transition point in the large $n$ limit by $\alpha_k$. The basic mathematical tool employed for bounding $\alpha_k$ from above is a probabilistic technique known as the first moment method. This method makes use of Markov's inequality: let $X$ be a nonnegative integer random variable and let $E[X]$ be the expectation of $X$, then $P_r[X \geq 1] \leq E[X]$. In our case Markov's inequality is applied on a sequence of random variables $X = X_n, n = 0, 1, \ldots$, that depend on certain control parameters. If one finds a condition on the control parameters that forces $E[X]$ to approach zero as $n$ approaches infinity, then the probability of $X$ being nonzero also vanishes in the large $n$ limit as long as that condition holds. Despite its simplicity, the first moment method is a powerful tool that quickly provides us with a condition (though most often not the tightest possible) for proving that asymptotically a random variable is almost certainly zero.

The connection of the first moment method with the satisfiability threshold conjecture was observed by a number of researchers, including Franco and Paull [22], Simon et al. [47] and Chvátal and Szemerédi [12]. Let $\phi$ be a random 3-SAT formula on $n$ variables generated according to $G_{m,m}$ and let $A_n(\phi)$—or simply $A_n$ if $\phi$ is implied by the context—be the random set consisting of the truth assignments that satisfy $\phi$. The probability that a truth assignment satisfies a single clause is $7/8$, so given $2^n$ possible truth assignments, $E[|A_n|] = 2^n (7/8)^m$. Since $P_r[\phi \text{ is satisfiable}] = P_r[|A_n| \geq 1]$, from Markov’s inequality it follows that

$$P_r[\phi \text{ is satisfiable}] \leq 2^n \left(\frac{7}{8}\right)^{\alpha_n}.$$  

If by $\alpha_M$ we denote the exact solution of the equation $2(7/8)^{\alpha} = 1$ (so that $\alpha_M = \log 2 / \log(8/7) \approx 5.19$), then we observe that under the condition $\alpha > \alpha_M$ the right-hand side of eq. (1) tends to zero. This establishes the value $\alpha_M$ as an upper bound for the critical value $\alpha_3$.

It is perhaps instructive at this point to provide the Markov inequality computations for model $G_p$ as an example of the difference in accuracy that can be obtained using various random models. In $G_p$, the probability that a truth assignment satisfies a random formula is the probability that none of the $\binom{n}{3}$ clauses violated by the assignment are part of the formula, or $(1 - p)^{\binom{n}{3}}$. Let us set $p = (6\alpha)/(8n^2)s$, so that for large $n$ the mean number of clauses in $\phi$ is $\alpha n$. Note that for such a choice of selection probability, it holds that if the event “$\phi$ is satisfiable” has a vanishingly small probability in the $G_p$ model, the probability
of this event is also small in $G_m$ and $G_{m,m}$ for $m = \alpha n$, as well as in $G_{3,\alpha n}$. By Markov’s inequality in $G_p$ we have

$$\Pr[\phi \text{ is satisfiable}] \leq \mathbb{E}[|A_n|] = 2^n \left(1 - p(n)\right)^{\binom{n}{\alpha}} = 2^n \left(1 - \frac{6\alpha}{8n^2}\right)^{\binom{n}{\alpha}},$$

so asymptotically, $\Pr[\phi \text{ is satisfiable}] \leq 2^n e^{-\alpha n/8}$. This leads to the inequality $\alpha_3 < 8 \log 2 \approx 5.545$, a weaker one than for $G_{m,m}$. Equations (1) and (2) provide a simple demonstration of a frequently occurring tradeoff among the various probabilistic models: accuracy of results vs. ease of handling complicated situations, such as the computation of the probability of conjunctions of events.

The first observation that the inequality $\alpha_3 < 5.19$ is not the best one possible came from Broder, Frieze, and Upfal [7], who pointed out that the condition $\alpha > \alpha_M - 10^{-7}$ is sufficient to guarantee that $\Pr[\phi \text{ is satisfiable}]$ tends to zero. El Maftouhi and Fernandez de la Vega [19] obtained a further improvement, by showing that the condition can be relaxed to $\alpha > 5.08$. Then Kamath et al. [33] obtained the improved condition $\alpha > 4.758$ using a numerical computation while also giving an analytical proof of the condition $\alpha > 4.87$. Using a refinement of Markov’s inequality based on the definition of a restricted class of satisfying truth assignments, Kirousis, Kranakis, and Krizanc [38] proved an upper bound value $\alpha > 4.667$. Using the same class of satisfying truth assignments, after more accurate but lengthier computations, Dubois and Boufkhad [16] independently obtained the upper bound 4.642. Also, Kirousis et al. [40] give the bound 4.602 by what they call “the method of local maxima.” Later, Janson, Stamatiou, and Vamvakari [32], lowered this value to 4.596 through two different approaches: by viewing a formula as a physical spin system (using language similar to that of Kobe [42]) and taking advantage of techniques from statistical physics to compute an asymptotic expression for its energy, and by obtaining an improved upper bound to the Rogers-Szegő polynomials. In Zito’s doctoral thesis [50] the upper bound was further improved to about 4.58 while Kaporis et al. [37] obtained the value 4.571 using a new upper bound for the $q$-binomial coefficients obtained in [41]. Finally, Dubois, Boufkhad, and Mandler [18, 17] gave an upper bound of 4.506 using an approach involving formulas with a “typical” number of appearances of signed occurrences of their variables.

For general $k$, Franco and Paull [22] used the first moment method and derived an upper bound for the value of the satisfiability threshold of $k$-SAT equal to $2^k \log 2$, while the same derivation was also observed by Simon et al. [47] and Chvátal and Szemerédi [12]. Kirousis, Kranakis, Krizanc and Stamatiou [40] and, independently, Dubois and Boufkhad [16] gave techniques that improved this general upper bound without, however, improving the leading term that in both approaches is equal to $2^k \log 2$.

On the lower bound side, Chao and Franco [8, 9] were the first to analyze the asymptotic behavior of algorithms that apply a heuristic in order to iteratively assign a truth value to all the variables of a formula. If the heuristic is sure
6 The Satisfiability Threshold Conjecture

to succeed at a given value of \( \alpha \), this clearly provides a lower bound on the threshold. One of the algorithms they analyzed applied the \textit{unit clause} heuristic defined in Cocco et al. [13]. Using a technique relying on differential equations in order to model the workings of their algorithm, they showed that the algorithm succeeds with positive probability (but \textit{not} necessarily with high probability) for clause-to-variable ratio less than 2.9. Following this, the first lower bound for 3-SAT was established by Franco [21], who analyzed an algorithm that satisfies only literals whose complements do not appear in the formula (\textit{pure literals}). He showed that for \( \alpha < 1 \), the algorithm succeeds almost certainly—meaning with probability approaching one in the large \( n \) limit—in finding a satisfying truth assignment to all the variables. Broder, Frieze and Upfal [7] then showed that the pure literal heuristic actually succeeds almost certainly in satisfying a formula if the ratio is smaller than 1.63. Frieze and Suen [24] improved the lower bound to 3.003 by analyzing the \textit{generalized unit clause} heuristic (GUC) with limited backtracking and showing that it succeeds almost certainly for ratios lower than 3.003, as discussed in Cocco et al. [13]. Finally, using the differential equations method developed by Wormald [49] for approximating the evolution of discrete random processes, Achlioptas [1] and Achlioptas and Sorkin [3] reached the values 3.143 and 3.26, respectively. They developed a framework for a special class of algorithms called \textit{myopic}, and showed that no algorithm in this class can succeed almost certainly in satisfying formulas with clause-to-variable ratios larger than 3.26. Recently, Kaporis, Kirousis, and Lalas [34, 35] analyzed a simple greedy heuristic using the methodology discussed in Kaporis et al. [36], where the literal that is selected to be satisfied at each step is the one with the maximum number of occurrences in the formula. They obtained the lower bound of 3.42. This was the first time that a heuristic making use of information related to the number of appearances of literals in a random formula (\textit{degree sequence}) has been analyzed. With a little more complicated greedy heuristics that at each step satisfy a literal with a large degree but whose negation has a small degree, a lower bound of more than 3.52 can be attained. This is currently the best value.

The best currently known \textit{general} lower bound for \( k \)-SAT, for any fixed value of \( k \), is given by a recent result by Achlioptas and Moore [2] who showed that \( \alpha_k \geq 2^k \log 2/2 - c \), for some constant \( c > 0 \) independent of \( k \). This result essentially bridged the asymptotic gap between the \( 2^k \log 2 \) general upper bound and the 1.817(\( 2^k/k \)) previously best general lower bound obtained by Frieze and Suen [24]. Moreover, Frieze and Wormald [25] showed that \( \alpha_k \) is asymptotic to \( 2^k \log 2 \) if \( k \) is a function of \( n \) and \( k - \log_2 n \to \infty \). Both results are the first successful efforts (to the best of our knowledge) in applying the \textit{second moment method} in order to prove a lower bound to the satisfiability threshold, something that previously was feasible only through the probabilistic analysis of satisfiability algorithms relying on specific heuristics for random formulas, as discussed above. Finally, using a technique known in physics as the \textit{replica method}, Monasson and Zecchina predict [45] that the asymptotic (in \( k \)) expression for the threshold is equal to \( 2^k \log 2 \), although their approach is not a rigorous one.
4 UPPER BOUND APPROACHES BASED ON THE HARMONIC MEAN

The first moment method is simple to apply, but does not lead to the best possible upper bounds for $k$-SAT. For values of the clause-to-variable ratio smaller than $\alpha_M$ as defined in the previous section, the expected number of satisfying truth assignments of a random formula tends to infinity, even though the empirical evidence suggests that most such formulas have no satisfying truth assignment at all. This is due to the fact that there exist very rare formulas that are satisfiable and have a large number of satisfying assignments. El Maftouhi and Fernandez de la Vega [19] and, independently, Kamath et al. [33] studied this situation in detail. They resorted to the harmonic mean formula, first introduced (or formalized) by Aldous [4] to address the problem.

Aldous’s result. Let $(B_i : i \in I)$ be a finite family of events in a probability space. For a permutation $\pi$ of $I$, call $(B_i)$ invariant under $\pi$ if
\[
\Pr[B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_r}] = \Pr[B_{\pi(i_1)} \cap B_{\pi(i_2)} \cap \cdots \cap B_{\pi(i_r)}]
\]
for all $\alpha \geq 1$ and $i_1, \ldots, i_r \in I$. Call the family $(B_i)$ transitive invariant if for each $i_1, i_2 \in I$ there exists $\pi$ such that $\pi(i_1) = i_2$ and $(B_i)$ is invariant under $\pi$. In particular, transitive invariance implies that $\Pr[B_i] = p$ is actually independent of $i$.

Let $N$ be the random variable counting the number of $B_i$’s that occur. Then, if $(B_i : i \in I)$ is a transitive invariant family of events (with $p = \Pr[B_i]$ independent of $i$),
\[
\Pr[\bigcup_{i \in I} B_i] = p \cdot |I| \cdot \mathbb{E}[N^{-1}|B_j]
\]
for any $j \in I$. The method gives a new expression for $\Pr[\phi$ is satisfiable$]$, if one interprets $B_i$ as the event “assignment $A_i$ satisfies $\phi$.” (Note that $|A_n|$, the number of truth assignments satisfying $\phi$, is denoted by $|\text{Mod}(F)|$ in El Maftouhi and Fernandez de la Vega [19] and $\#F$ in Kamath et al. [33].) Let $T_i$ be the set of formulas satisfied by the $i$th truth assignment when these assignments are placed in reverse lexicographic order, so that $T_1$ consists of those formulas satisfied by all variables set to TRUE. In that case, letting $j = 1$ without loss of generality, the following is a restatement of Aldous’s result in the context of 3-SAT formulas:
\[
\Pr[\phi$ is satisfiable$] = \mathbb{E}[|A_n|] \times \sum_{\psi \in T_1} \frac{1}{|A_n(\psi)| \cdot |T_1|}.
\]

Notice that an expression equivalent to the equation above is the following (this is the one proved explicitly in El Maftouhi and Fernandez de la Vega [19, eq. (1)]):
\[
\mathbb{E}_{\phi \in \text{SAT}}[|A_n|] = \frac{|T_1|}{\sum_{\psi \in T_1} |A_n(\psi)| \cdot |T_1|}. 
\]
where $E_{\phi \in \text{SAT}}[|\mathcal{A}_n|]$ is the expectation of $|\mathcal{A}_n|$ with respect to all satisfiable formulas on $n$ variables and $m$ clauses. The authors in El Maftouhi and Fernandez de la Vega [19] prove that it is possible to define a class of formulas $T_1^* \subseteq T_1$ of size at least $(1 - 2^{-\delta n})|T_1|$, where $\delta$ is a constant, such that each formula in $T_1^*$ has at least $2^{\delta n}$ satisfying truth assignments. As we will see in subsection 4.1, this implies $E_{\phi \in \text{SAT}}[|\mathcal{A}_n|] \geq 2^{\delta n - 1}$. Therefore, the probability that a random 3-SAT formula is satisfiable is at most

$$2^n \left( \frac{7}{8} \right)^{\alpha n} 2^{-\delta n}.$$

The authors set $\alpha = 5.08$, and using a simple random experiment they find a class of formulas $T_1^*$ satisfying the conditions above when $\delta = 0.02137$. The satisfiability probability goes to zero asymptotically for these values, establishing the improved bound $\alpha_3 < 5.08$. The different quality of the bounds derived in El Maftouhi and Fernandez de la Vega [19] and Kamath et al. [33] is due not only to the use of coarse upper bounds rather than exact asymptotics in El Maftouhi and Fernandez de la Vega [19] for estimating the proportion of “interesting” formulas with a particular structure—it can in fact be proven that the difference between the two is vanishingly small—but also to the different experiment used to count this proportion. In the following sections we report, briefly, the results in the two papers. The careful reader will be able to pick up the similarities and the differences in the two approaches.

### 4.1 ACCOUNTING FOR RARE FORMULAS WITH MANY SATISFYING TRUTH ASSIGNMENTS: DISPENSABLE VARIABLES

El Maftouhi and Fernandez de la Vega [19] define the subset $T_1^*$ of $T_1$ such that $|T_1^*| \geq (1 - 2^{-\delta n})|T_1|$ and all formulas in $T_1^*$ have at least $2^{\delta n}$ satisfying assignments. Notice that this can be rewritten as $\Pr[|\mathcal{A}_n| \geq 2^{\delta n} | \phi \in T_1] \geq 1 - 2^{-\delta n}$. Since $|\mathcal{A}_n| \geq 1$ for any $\phi \in T_1$, one can therefore write:

$$\sum_{\psi \in T_1} \left\{ \frac{1}{|\mathcal{A}_n(\psi)|} \right\} = \sum_{\psi \in T_1^*} \left\{ \frac{1}{|\mathcal{A}_n(\psi)|} \right\} + \sum_{\psi \in T_1 \setminus T_1^*} \left\{ \frac{1}{|\mathcal{A}_n(\psi)|} \right\} \leq \frac{|T_1|}{|T_1|^2} + \frac{|T_1| - |T_1^*|}{2^{\frac{\alpha n}{2}}}.$$

Therefore $E_{\phi \in \text{SAT}}[|\mathcal{A}_n|] \geq \frac{|T_1|}{2^{\frac{\alpha n}{2}}} = 2^{\delta n - 1}$.

In order to describe how $T_1^*$ is defined, let $C_i = C_i(\psi)$ be the set of clauses in $\psi$ containing exactly $i$ positive literals. We first estimate $|C_i|$ under the assumption that $\psi \in T_1$. Notice that no formula in $T_1$ can contain a clause with only negated variables, therefore, $|C_0| = 0$. Furthermore, for formulas in $T_1$ with $n$ variables and $\alpha n$ clauses, it is fairly easy to compute the asymptotic distribution of the
formulas with \( |C_i| = m_i \) for each \( i \in \{1, 2, 3\} \) (where \( \alpha n = m_1 + m_2 + m_3 \)):

\[
\Pr[m_1, m_2, m_3] = \frac{(m_{\alpha n}) \binom{n}{2}^{m_1} \binom{n}{3}^{m_2} \binom{n}{3}^{m_3}}{\left(\frac{2n}{3} - \binom{n}{3}\right)^{\alpha n}}.
\]

Using Stirling’s approximation for the various factorials involved and setting \( \gamma_i = m_i/n \) it is easy to prove that \( \Pr[m_1, m_2, m_3]^{1/n} \) is asymptotic to \( (6\alpha/7)^{\alpha} / (2\gamma_1)^{\gamma_1}(2\gamma_2)^{\gamma_2}(6\gamma_3)^{\gamma_3} \), assuming all \( m_i \)'s tend to infinity. Considered as a function of \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), this expression reaches its maximum (equal to one) for \( \gamma_1 = \gamma_2 = \frac{3\alpha}{2} \) and \( \gamma_3 = \frac{\alpha}{2} \). Now let \( T_1^* \subseteq T_1 \) be the set of all those formulas in \( T_1 \) with \( \gamma_1 \leq 2.37, \gamma_2 \leq 2.37 \) and \( \gamma_3 \leq 0.87 \) (recall that \( \alpha = 5.08 \) for all formulas in \( T_1 \)). It may be shown from the asymptotic probability expression [19] that these inequalities hold with probability greater than \( 1 - 2^{-0.02137n} \), implying that \( |T_1^*| > (1 - 2^{-0.02137n})|T_1| \).

To prove that 5.08 is an upper bound to the satisfiability threshold, following the reasoning given earlier, it remains to demonstrate that the formulas in \( T_1^* \) have, with sufficiently high probability, at least \( 2^{0.02137n} \) satisfying truth assignments. To this end, the authors introduce the notion of dispensable variables. Given a formula \( \phi \), a truth assignment \( A \) that satisfies \( \phi \) and a set \( D \) consisting of certain variables taking the values dictated by \( A \), we call \( D \) a set of dispensable variables if its elements can be set in any arbitrary way and still result in the truth assignment satisfying \( \phi \). Let \( D(\phi) \) be the set of dispensable variables in \( \phi \) with respect to the assignment that sets all variables to TRUE. Clearly \( |A_n(\phi)| \geq |D(\phi)| \), so it is then sufficient to show that for all \( \phi \in T_1^* \), \( |D(\phi)| \geq 0.02137n \) with high probability. The authors do this by analyzing the size of the set of dispensable variables returned by the following “greedy” algorithm:

1. Take all clauses in \( C_1 \), and call \( I_1 \) the set of all positive literals in these clauses. These are known as isolated literals. Let \( n_1 = |I_1| \).
2. Take all clauses in \( C_2 \) whose two positive literals are both absent from \( I_1 \), and for each such clause select at random one of its two positive literals. Call \( I_2 \) the set of all such literals. Set \( J_2 = I_1 \cup I_2 \). Let \( n_2 = |I_2| \), so that \( |J_2| = n_1 + n_2 \).
3. Take all clauses in \( C_3 \) whose three positive literals are all absent from \( J_2 \), and for each such clause select at random one literal. Call \( I_3 \) the set of all such literals. Set \( J_3 = J_2 \cup I_3 \). Let \( n_3 = |I_3| \), so that \( |J_3| = n_1 + n_2 + n_3 \).

One may readily verify that all variables “not” represented in \( J_3 \) form a set of dispensable variables, so \( |J_3| \) needs to be bounded from above. For the range of values of \( \gamma_i \) that defines \( T_1^* \), an estimate on \( |J_3| \) is obtained by finding upper bounds on: \( n_1 \); \( n_2 \) conditioned on \( n_1 \); and \( n_3 \) conditioned on \( n_2 \) and \( n_1 \).

In order to estimate \( n_3 \), the authors resort to the occupancy problem. In this problem, one throws \( \mu n \) balls (\( \mu \) is a constant) uniformly at random into \( n \) boxes and asks for the distribution of the random variable \( Y \) that counts the number
The Satisfiability Threshold Conjecture

of non-empty boxes. Then for any \( \epsilon > 0 \), \( r = r(\epsilon) = (1 + \epsilon)(1 - e^{-\frac{\epsilon}{1 + \epsilon}}) \) and \( s = s(\epsilon) = 1 - r(\epsilon) \), the following is established:

\[
\frac{1}{n} \log \Pr \left[ Y \geq \lceil r(\epsilon) n \rceil \right] \leq \left( 1 - o(1) \right) \log \left( \frac{(s + \epsilon)^{s+\epsilon}(1 + \epsilon)^{n-1-\epsilon}}{s^s} \right).
\]

As \( n_1 \) can be viewed as the number of non-empty boxes that result from the random placement of \( \gamma_1 n \) balls into \( n \) boxes, we have that \( Y = n_1 \) and \( \mu = \gamma_1 \). Setting \( \epsilon = 0.062 \) and exponentiating both sides of the inequality above, we obtain for \( \gamma_1 \leq 2.37 \) that

\[
\Pr [n_1 \leq 0.94800n] \geq 1 - e^{-0.01513n}.
\]

Now define \( m'_2 \) as the number of clauses in \( C_2 \) (clauses with two positive literals) identified by the greedy algorithm as having both of their positive literals absent from \( I_1 \). This is binomially distributed, with number of trials \( 2n_1 \) and success probability \( (1 - n_1/n)(1 - (n_1 - 1)/n) \), where success means “absent from \( I_1 \).” Conditioning on \( n_1 \leq 0.94800n \), we can use the Chernoff bound on the upper tail of the binomial distribution, \( \Pr [B(m, p) \geq \beta mp] \leq (e^{\beta - 1}/\beta) ^{mp} \), setting \( \beta = 3.84 \) and \( mp = 0.006408n \) to obtain for \( \gamma_2 \leq 2.37 \) that \( \Pr [m'_2 \leq 0.02461n | n_1 \leq 0.94800n] \geq 1 - e^{-0.01490n} \). Since \( n_2 \leq m'_2 \),

\[
\Pr [n_2 \leq 0.02461n | n_1 \leq 0.94800n] \geq 1 - e^{-0.01490n}.
\]

Similarly, define \( m'_3 \) as the number of clauses in \( C_3 \) identified by the greedy algorithm as having all of their literals absent from \( J_2 \). Again bounding the tail of the relevant binomial distribution, we obtain for \( \gamma_3 \leq 0.87 \) that \( \Pr [m'_3 \leq 0.00356n | n_1 + n_2 \leq 0.97261n] \geq 1 - e^{-0.0153n} \). Since \( n_3 \leq m'_3 \),

\[
\Pr [n_3 \leq 0.00356n | n_1 + n_2 \leq 0.97261n] \geq 1 - e^{-0.0153n}.
\]

Finally, multiplying together (3), (4) and (5) we may verify that for sufficiently large \( n \), \( \Pr [n_1 + n_2 + n_3 \leq 0.97617n] \geq 1 - e^{-0.01481n} = 1 - 2^{-0.02137n} \). Thus, with probability at least \( 1 - 2^{-0.02137n} \), \( |D(\phi)| \geq 0.02383n > 0.02137n \) for \( \phi \in T_1^* \).

4.2 SHARPER ESTIMATE OF OCCUPANCY PROBABILITIES: INDEPENDENT VARIABLES

Kamath et al. [33] performed a similar investigation of the structure of the typical \( \phi \in T_1^* \). A variable \( x \) is said to cover a clause \( C \) if \( x \) occurs unnegated in \( C \)—that is, as a positive literal. For instance, in the formula below (which does not belong to \( T_1^* \)), represented by the sequence of sets of literals forming individual clauses in it,

\[
\phi(x_1, x_2, x_3, x_4, x_5) =
\begin{align*}
&\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \\
&\{x_2, x_3, x_4\}, \{x_1, x_4, x_5\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_5\}, \{x_3, x_4, x_5\},
\end{align*}
\]

\( C_1 \)
A set $X$ of variables and the clauses in $C_1$, $C_2$, $C_3$ that it covers. Broken lines denote covered regions; solid lines represent some of the specific clauses covered by variables in $I_1$.

The variable $x_1$ covers the clauses:

\[
\{x_1, x_2, x_3\}, \{x_1, x_4, x_5\}, \{x_1, x_2, x_5\}, \{x_1, x_4, x_4\}, \{x_1, x_3, x_5\}.\]

A set of variables $V$ covers a set of clauses if every one of these clauses is covered by at least one variable in $V$. Such a variable set is called a cover. Obviously, a random formula $\phi \in \mathcal{T}_1$ has a “trivial” cover, namely the set consisting of all the $n$ variables. However, it is possible that there exists a smaller cover than the trivial one. For instance, Figure 1 shows a formula in $\mathcal{T}_1$ with the sets of clauses in $C_1$, $C_2$, $C_3$, the set $X$ containing all variables, a set $I_1$ of variables covering $C_1$, a second set of variables (from the set $R$ of remaining variables) needed to cover the uncovered portions of $C_2$ and $C_3$, and the remaining independent set $I$. The set $X \setminus I$ in the figure is an example of a cover for all the clauses of the formula that is smaller than the trivial one. Therefore, setting the variables in $X \setminus I$ to TRUE is sufficient to satisfy the formula. Since all the
variables in the independent set $I$ can be set arbitrarily, if $\phi$ has a cover of size $s$ then $|A_n(\phi)| \geq 2^n - s$.

To estimate $E_{\phi \in \text{SAT}}[|A_n|] = |T_1|/\sum_{\psi \in T_1} 1/|A_n(\psi)|$, we partition $T_1$ into “slices” containing formulas with minimal cover size $s$, and use:

$$\sum_{\psi \in T_1} \frac{1}{|A_n(\psi)| \cdot |T_1|} = \sum_{s=1}^{n} \frac{1}{\sum_{\psi \in T_1 : |\text{cover}(\psi)| = s} |A_n(\psi)|} \cdot \frac{1}{|T_1|} \leq \sum_{s=1}^{n} \sum_{\psi \in T_1 : |\text{cover}(\psi)| = s} \frac{\Pr[|\text{cover}(\psi)| = s]}{|A_n(\psi)|} \leq \sum_{s=1}^{n} \sum_{\psi \in T_1 : |\text{cover}(\psi)| = s} 2^{s-n} \Pr[|\text{cover}(\psi)| = s].$$

The problem thus reduces to one of estimating, as accurately as possible, the probability that an arbitrary formula in $T_1$ has minimal cover size $s$. As in subsection 4.1, this is done using asymptotic expressions for binomial tails and occupancy probabilities.

1. First fix the size $m_1$ of $C_1$, the set of clauses containing a single unnegated variable. The probability $\Pr[u]$ that this number is within $u$ times its mean can be shown to be close to 1 using bounds on binomial tails.
2. Then determine the set $I_1$ of isolated variables ($X \setminus R$ in fig. 1). Conditioning on $m_1$ being within $u$ of its mean, the probability $\Pr[v|u]$ that the size $n_1$ of $I_1$ is within $v$ times its mean is estimated using occupancy asymptotics (again we are throwing clauses “into” variables: the empty bins correspond to the variables in $X \setminus I_1$).
3. Next, compute the number of clauses in $C_2 \cup C_3$ that are “not” covered by $I_1$, conditioned on $n_1$ and $m_1$. The set of these clauses is denoted by $U$ in figure 1. The probability that this number is within $w$ times its mean, $\Pr[w|u,v]$, is again a binomial tail.
4. Finally, bound the size of the variable set needed to cover $U$.

The improvement in Kamath et al. [33] comes from estimating the size of $U$, and from adding to the cover only those variables needed to cover $U$ rather than the whole set $I_2 \cup I_3$ (as done in the other paper, in the second and third selection step).

5 SPECIAL CLASSES OF SATISFYING TRUTH ASSIGNMENTS

The next improvements resulted from a different kind of exploitation of rare formulas with many satisfying truth assignments.
Since the main disadvantage of Markov’s inequality can be attributed to rare formulas having a large number of satisfying truth assignments, a plausible approach for improvement is to use in the inequality “not” the expected cardinality of the set of satisfying truth assignments of a random formula, but the expected cardinality of a “smaller” set. Of course one needs to prove that the expectation of the new random set still provides an upper bound on the probability that $\phi$ is satisfiable.

This idea was introduced by Kirousis, Kranakis, and Krizanc [38] (“single flips”) and independently by Dubois and Boufkhad [16]. In this section we describe these approaches and several others derived from them, and we report the resulting improvements on the estimate of $\alpha_3$. With regard to the techniques described in sections 5.1 and 5.2 in particular, we should point out in advance the following important difference between them. The technique described in section 5.1 approximates $\alpha_3$ by computing “an upper bound” on the expected cardinality of a special class of satisfying truth assignments, employing a simple correlation inequality that bounds from above the probability that some dependent events hold simultaneously. On the other hand, the technique described in section 5.2 results in a slightly better approximation of $\alpha_3$ by computing “exactly” the expected cardinality of the same class of satisfying truth assignments. The former method, however, is much simpler to apply and easily generalizes to $k$-SAT random formulas for any $k > 3$ while the latter is complicated and cannot be readily applied to $k$-SAT in general.

The following formula

$$\phi(x_1, x_2, x_3, x_4) = \{x_1, \overline{x}_2, x_4\}, \{\overline{x}_1, x_2, \overline{x}_4\}, \{x_1, x_3, \overline{x}_4\}, \{\overline{x}_1, x_2, x_3\}$$

whose satisfying assignments are

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>FALSE</td>
<td>FALSE</td>
<td>FALSE</td>
<td>FALSE</td>
</tr>
<tr>
<td>$A_2$</td>
<td>FALSE</td>
<td>FALSE</td>
<td>FALSE</td>
<td>TRUE</td>
</tr>
<tr>
<td>$A_3$</td>
<td>FALSE</td>
<td>FALSE</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>$A_4$</td>
<td>FALSE</td>
<td>TRUE</td>
<td>FALSE</td>
<td>TRUE</td>
</tr>
<tr>
<td>$A_5$</td>
<td>TRUE</td>
<td>FALSE</td>
<td>FALSE</td>
<td>FALSE</td>
</tr>
<tr>
<td>$A_6$</td>
<td>TRUE</td>
<td>FALSE</td>
<td>FALSE</td>
<td>TRUE</td>
</tr>
<tr>
<td>$A_7$</td>
<td>TRUE</td>
<td>TRUE</td>
<td>FALSE</td>
<td>FALSE</td>
</tr>
<tr>
<td>$A_8$</td>
<td>TRUE</td>
<td>TRUE</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
</tbody>
</table>

will be used as an example in the next subsection, to convey a better understanding of the various results.

### 5.1 SINGLE FLIPS

The strategy described here was introduced in Kirousis et al. [38]. In what follows it may be convenient to identify classes of truth assignments on $n$ variables with
sets of lexicographically ordered sequences over a two-letter alphabet (say the numbers 0 and 1 with \(0 < 1\)).

Given a random formula \(\phi\), the set \(A^1_n\) is defined as the class of truth assignments \(A\) such that the following two conditions hold: (1) \(A\) satisfies \(\phi\), and (2) any assignment obtained from \(A\) by changing exactly one FALSE value to TRUE does not satisfy \(\phi\). Such a change is called a *single flip* and will be denoted by \(sf\). The truth assignment that results from the flip will be denoted by \(A_{sf}\).

The class \(A^1_n\) contains the elements of \(A_n\) that are local maxima with respect to single flips. In other words, a truth assignment belongs to \(A^1_n\) if it satisfies \(\phi\) and if no other possible satisfying truth assignment can be obtained from it by changing a single false value to true—by performing all possible single flips in isolation.

In our example formula, the first truth assignment \(A_1\) does not belong to \(A^1_n\): if we change the value assigned to \(x_1\) the resulting truth assignment is still in \(A_n\). However, the fourth truth assignment \(A_4\) does belong to \(A^1_n\): changing the value assigned to \(x_1\) produces the assignment \(A' \not\in A_n\) where all variables except \(x_3\) are set to true, and changing the value of \(x_3\) similarly leads to the \(\phi\) not being satisfied. Since all possible transformations changing a false value result in truth assignments that do not satisfy \(\phi\), the assignment \(A_4\) is in \(A^1_n\). It is easy to verify that the set \(A^1_n\) for \(\phi\) is formed by the assignments \(A_3, A_4, A_6,\) and \(A_8\).

Since \(A^1_n \subseteq A_n\), \(E[|A^1_n|] \leq E[|A_n|]\). Thus, to relax Markov’s inequality we need only establish that \(\Pr[\phi \text{ is satisfiable}] \leq E[|A^1_n|]\). This can easily be seen as follows. Let \(I_\phi\) be the random indicator for the property “\(\phi\) is satisfiable.” Clearly \(I_\phi \leq |A^1_n|\). If we now write

\[
\Pr[\phi \text{ is satisfiable}] = \sum_\phi I_\phi \Pr[\phi]
\]

then the desired inequality follows immediately. To exploit this technique one then needs to prove a bound on \(E[|A^1_n|]\), asymptotically smaller than \(2^n (7/8)^n\).

Using a correlation inequality to compute the probability that a single flip results in an assignment not satisfying \(\phi\), it can be proven that in the random formula model \(G_{m,m}\), the expected size of class \(A^1_n\) is at most \((7/8)^\alpha (2 - e^{-3\alpha/7} + o(1))^n\). Therefore, the unique positive solution of the equation \((7/8)^\alpha (2 - e^{-3\alpha/7}) = 1\) gives an upper bound for the satisfiability threshold critical value \(\alpha_3\). This solution is approximately 4.667.

If instead one uses the \(G_p\) ensemble, one avoids having to compute probabilities of conjunctions of dependent events. Applying Markov’s inequality then leads to the solution of the equation \(e^{-\alpha/8} (2 - e^{-3\alpha/7}) = 1\). This expression, however, gives an upper bound equal to 5.07—worse than the bound given for \(G_{m,m}\).
5.2 THE SET OF NEGATIVELY PRIME SOLUTIONS (NPS)

Independently from Kirousis et al., Dubois and Boufkhad introduced a class of satisfying truth assignments that they called negatively prime solutions (NPS) [16]. This class turns out to coincide with the class $A_{1n}$ described in Subsection 5.1.

Dubois and Boufkhad proved the following exact expression for the expected cardinality of NPS for $k$-SAT:

$$E[|NPS|] = \sum_{i=0}^{n} \sum_{j=i}^{m} 2^{-km} \binom{n}{i} \binom{m}{j} S_2(j,i) i! \left(\frac{k}{n}\right)^j (2^k - 1)^{m-j}$$

(6)

where $S_2(j,i)$ are the Stirling numbers of the second kind that count the number of ways of partitioning a set of $j$ elements into $i$ non-empty subsets. By way of a series of asymptotic manipulations, the authors then arrived at a closed-form upper bound for (6), showing that it converges to 0 for values of the clause-to-variable ratio greater than 4.642.

5.3 RESTRICTING FURTHER THE CLASS OF SATISFYING TRUTH ASSIGNMENTS: DOUBLE FLIPS

Kirousis et al. [40] define as a double flip the change of exactly two variables $x_i$ and $x_j$, with $i < j$, where $x_i$ is changed from FALSE to TRUE and $x_j$ from TRUE to FALSE. Notice that the restriction $i < j$ implies that a double flip always leads to a lexicographically “greater” assignment. Let $A_{df}$ denote the truth assignment that results from $A$ after the application of the double flip $df$. Let $A_{2}^{\neq} n$ be the set of truth assignments $A$ that have the following three properties: (1) $A$ satisfies $\phi$, (2) for all possible single flips $sf$ of $A$, $A^{sf}$ does not satisfy $\phi$, and (3) for all possible double flips $df$ of $A$, $A^{df}$ does not satisfy $\phi$.

It is proven in [40] that the following inequality holds:

$$\Pr[\phi \text{ is satisfiable}] \leq E[|A_{2}^{\neq} n|] = \left(\frac{7}{8}\right)^\alpha n \sum_{A \in S} \Pr[A \in A_{n} | A \in A_{n}] \cdot \Pr[A \in A_{2}^{\neq} | A \in A_{1n}],$$

(7)

where $S$ is the set of all $2^n$ possible truth assignments on the $n$ variables. As before, to get an upper bound on $\alpha_3$ it suffices to find the smallest possible value for the clause-to-variable ratio for which the right-hand side of this inequality tends to 0. In what follows, $sf(A)$ denotes the total number of possible single flips of $A$ (which is simply the number of variables assigned the value FALSE in $A$) and $df(A)$ denotes the total number of possible double flips.

It can be proven that $\Pr[A \in A_{1n} | A \in A_{n}]$ may be bounded from above by an expression of the form $P^{sf(A)}$, where $P$ depends only on the clause-to-variable ratio $\alpha$ (this is in fact the expression used to derive the improved upper bound in subsection 5.1). The hard part is computing the second probability,
16 The Satisfiability Threshold Conjecture

involving the realization of the double flip events conditional upon the realization of the single flip events. As it turns out, the computation of this probability in the model $G_{m,m}$ involves dependencies among events of a very complicated nature. In $G_p$, on the other hand, there are no dependencies ensuing from the fundamental requirement of $G_{m,m}$ that the size of the formula is fixed. The remaining dependencies are those arising from the fact that some of the double flip events involve double flips sharing a particular false variable. The computation of an upper bound to this probability was made possible by the use of a version of Suen’s correlation inequality [48] proved by Janson [31].

The bound has the form $Q^{df}(A)$ with $Q$ dependent on $n$ and $\alpha$. The reader may consult [40] for the derivation of this bound. Inequality (7) may then be rewritten as follows:

$$\Pr[\phi \text{ is satisfiable}] \leq 3m^{1/2}(7/8)^n \sum_{A \in S} P^{sf}(A) Q^{df}(A),$$

with the polynomial factor $3m^{1/2}$ arising due to the change from $G_{m,m}$ to $G_p$ (see Bollobás [5]). To complete the derivation of the improved bound, the authors noted the following combinatorial identity, which can be proven by induction on $n$:

$$\sum_{A \in S} P^{sf}(A) Q^{df}(A) = \sum_{i=0}^{n} \left[\begin{array}{l}n \\ i \end{array}\right] P^i (1 + PQ^{i/2}),$$

where $\left[\begin{array}{l}n \\ i \end{array}\right]_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{i-1})}{(q^i - 1)(q^i - q) \cdots (q^i - q^{i-1})}$ are the $q$-binomial or Gaussian coefficients [26] for $0 \leq i \leq n$ and $q \neq 1$. The right-hand expression in eq. (9) is also known as the Rogers-Szegő polynomial $F_{n,Q}(P)$, and leads to the inequality

$$\sum_{A \in S} P^{sf}(A) Q^{df}(A) \leq \prod_{i=0}^{n-1} (1 + PQ^{i/2}), \quad 0 \leq P^2 \leq Q \leq 1.$$ 

Equation (8) thus becomes

$$\Pr[\phi \text{ is satisfiable}] \leq 3m^{1/2}(7/8)^n \prod_{i=0}^{n-1} (1 + PQ^{i/2}).$$

The product on the right-hand side can be estimated by making use of hypergeometric series (see also Gasper and Rahman [26]) and an inequality derived in Kirousis [39], ultimately leading to the inequality $\alpha_3 < 4.602$.

5.4 OCCUPANCY BOUNDS AND Q-BINOMIAL COEFFICIENTS

As we have just seen, a key step in the improvement of the upper bound to 4.602 is the derivation of an upper bound to the sum in eq. (8) through its connection with the $q$-binomial coefficients. This bound can be improved further: two approaches have been given by Janson, Stamatiou, and Vamvakari [32]. Both
result in the same value, namely $\alpha_3 < 4.596$. However, these approaches are interesting in their own right. The first approach links the problem of determining upper bounds to the satisfiability threshold with the study of Ising spin systems in statistical mechanics (see also Kobe [42]), while the second approach links the problem with the branch of mathematics dealing with $q$-hypergeometric series and their generating functions.

More specifically, in the first approach the sum in eq. (8) is written as

$$
\sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} \exp \left( a \sum_{i=1}^{n} \varepsilon_i + \frac{b}{n} \sum_{1 \leq i < j \leq n} \varepsilon_i (1 - \varepsilon_j) \right),
$$

(12)

with the outer sum ranging over all the $2^n$ sequences $\varepsilon_1, \ldots, \varepsilon_n$ of 0’s and 1’s, each of them coding a truth assignment $A$ with 0 and 1 representing TRUE and FALSE respectively. This sum is indeed equal to the sum of eq. (8) when $a = \log(P)$ and $b = n \log(Q)$.

The expression in eq. (12) enables the application of an optimization technique common in statistical physics, resulting in an asymptotic expression for the sum. This particular form of the sum is precisely the partition function $Z = \sum_{\varepsilon_1, \ldots, \varepsilon_n} \exp(-\beta H)$ for a system with $n$ spin sites, each having a spin value $\varepsilon_i \in \{0,1\}$ and with an energy function equal to $H = -a \sum_{i=1}^{n} \varepsilon_i - \frac{b}{n} \sum_{1 \leq i < j \leq n} \varepsilon_i (1 - \varepsilon_j)$, in units where the inverse temperature $\beta = 1$. The first term in $H$ corresponds to an external field acting on all the spins of the system, the second to an interaction acting between arbitrary pairs of spin sites with the left site having spin 1 and the right site having spin 0. The energy function can easily be rewritten into a more conventional form:

$$
H = -\frac{a}{2} n - \frac{b}{8} (n-1) + \sum_{i=1}^{n} \left( -\frac{a}{2} + \frac{b}{2n} \frac{(n+1)/2}{2n} \right) s_i + \frac{b}{4n} \sum_{i<j} s_is_j.
$$

The value $H$ may be interpreted in statistical physics as the energy function for a mean-field Ising model with an inhomogeneous (linear) external field. Ultimately, this leads to an asymptotic expression for the partition function, and an estimate for the sum in eq. (8), resulting in $\alpha_3 < 4.596$.

In the second approach, a sharp upper bound is derived for the sum in eq. (8). Recall that it is equal to the Rogers-Szegö polynomial $F_{n,Q}(P)$. Then, using the Eulerian generating function and a technique described in Lemma 8.1 of Odlyzko [46], the following upper bound is obtained (see Janson et al. [32]) for any $t$, $0 < t < \min(1,1/P)$:

$$
F_{n,Q}(P) \leq t^{-n} \exp \left[ -\frac{1}{\log Q} \left( \text{Li}_2(tP) + \text{Li}_2(t) + \text{Li}_2(Q^n) - \text{Li}_2(Q) \right) \right] \frac{1}{(1-t)(1-tP)}
$$
where $L_2(y) = \text{dilog}(1-y) = \text{Polylog}(2,y) = \sum_{i \geq 1} \frac{y^i}{i}$ is the dilogarithm function. By finding the value of $t$ that optimizes this upper bound and plugging it into eq. (8), we obtain precisely the same upper bound as in the first approach, $\alpha_3 < 4.596$.

5.5 BALLS AND BINS

The calculations described in the previous sections are useful, but limited. The idea of single flips, although leading to a good improvement of the upper bound, only takes into account a very limited range of locality around a given satisfying assignment. The results described in subsection 5.3 exploit wider locality ranges, but because of the complexity of the resulting numerical expressions, the authors were forced to use weak bounds on $\Pr[A \in A_{1n}^1 | A \in C_n]$ and the overall $E[|A_{1n}^2|]$. Finally, the two approaches in subsection 5.4 reached the limits of what can be exploited from the upper bound shown in eq. (8). In order to obtain further improvements, one must step back from the derivation of eq. (8) and attempt to realize improvements on the probabilities involved.

Kaporis et al. [37] have achieved an improved bound of 4.571 using sharp estimates for certain probabilities related to the classical occupancy problem that we have seen in subsection 4.1. For a given satisfying assignment $A$, the probability that no single flip satisfies $\phi$ is best computed (up to polynomial factors) by noticing the following “structural” condition imposed on $\phi$ by the event in question: for each variable $x$ set to false in $A$, the formula must contain a critical clause $\{x, \ell_1, \ell_2\}$ with $A(\ell_1) = A(\ell_2) = \text{false}$.

If a satisfying assignment $A$ sets $j$ variables to false, no single flip satisfies $\phi$ when: (1) some $l \geq j$ clauses out of $m = \alpha n$ are critical (the remaining ones being consistent with $A_{sf}$), and (2) these $l$ clauses can be seen as a sequence of balls that are dropped into $j$ distinct bins (corresponding to different single flips) in a way that leaves no bin completely empty.

Asymptotic estimates on the occupancy probability that the $l$ critical clauses indeed cover all possible $j$ single flips [33], as well as a change of models from $G_{m,m}$ to $G_m$ in order to be able to formulate our problem in the balls and bins framework, lead to a sharper bound of $\Pr[A \in A_{1n}^1 | A \in C_n]$. Note that Zito [50] has performed a similar analysis using coupon collector probabilities instead, deriving a bound of about 4.58. The analysis in Kaporis et al. [37] improves on the previous results for another reason as well: the overall bound on $E[|A_{1n}^2|]$ is tightened by means of a more direct estimate of the $q$-binomial coefficient involved. Using simple generating function inequalities (and elementary calculus) it is possible to bound the term $\binom{n}{i}$, directly and avoid the use of Rogers-Szegő polynomials.

Finally, to obtain their results, the authors had to establish a relationship between the probabilities of the event, $A \in A_{1n}^2 | A \in A_{1n}^1$, in the models $G_{m,m}$ and $G_p$. Using results described in Bollobás [5, Chap. II], it is easy to prove the desired relationship for unconditional events when the average length of a
random formula constructed in the model $\mathcal{G}_p$ equals $m$. However, the conditioning here may bias the expected length of the formula to higher values. The authors have shown how to adjust $p$ appropriately so as to obtain the bound above.

6 TYPICAL FORMULAS

In the last section, all the techniques work by placing restrictions on the assignments (semantics, in some sense) satisfying a formula and for which the expectation, which is required by the first moment method, is computed. As mentioned above, the application of this technique to the kinds of assignment restrictions described in subsections 5.1, 5.2, and 5.3 results provably in no further upper bound improvement.

Dubois, Boufkhad, and Mandler [18, 17] consider random formulas with the special characteristic that the numbers of appearances of their literals fall within certain ranges that are “typical” for randomly generated formulas. In this way, they are able to disallow the rare formulas that seem to prevent Markov’s inequality from giving an upper bound close to the experimentally determined value. By computing the expected number of negative prime solutions for these formulas only, making use of the model $\mathcal{G}_{3,\alpha_n}$, they achieve an upper bound improvement. In contrast to the semantic methods that rely on restricting the set of truth assignments taking part in the application of the first moment method, this approach can be characterized as syntactic: it focuses on restricting the form or syntax of the set of formulas participating in the first moment method calculations. However, the approach also limits the possible truth assignments using the restricted sets defined in Sections 5.1 and 5.2. Without going into detail, Dubois, Boufkhad and Mandler give an expression for the expected number of negative prime solutions for these formulas. In so doing, they obtain $\alpha_3 < 4.506$, the best upper bound to date for the location of the random 3-SAT threshold.

ACKNOWLEDGMENTS

We would like to thank the editors as well as the anonymous referee for their invaluable contribution to the improvement of the presentation in the paper. This work has been supported by the University of Patras Research Committee (Project C. Carathéodory no. 20445) and by the EU within the 6th Framework Programme under contract 001907 (DELIS).

REFERENCES

The Satisfiability Threshold Conjecture


The Satisfiability Threshold Conjecture


