DELIS-TR-0216

Structure and Complexity of Extreme Nash Equilibria

M. Gairing and T. Löcking and M. Mavronicolas and B. Monien and P. Spirakis

2005
Structure and complexity of extreme Nash equilibria

M. Gairing\textsuperscript{a, *}, T. Lücking\textsuperscript{a}, M. Mavronicolas\textsuperscript{b}, B. Monien\textsuperscript{a}, P. Spirakis\textsuperscript{c}

\textsuperscript{a}University of Paderborn, Paderborn, Germany
\textsuperscript{b}University of Cyprus, Nicosia, Cyprus
\textsuperscript{c}Computer Technology Institute & University of Patras, Patras, Greece

Abstract

We study extreme Nash equilibria in the context of a selfish routing game. Specifically, we assume a collection of \( n \) users, each employing a mixed strategy, which is a probability distribution over \( m \) parallel identical links, to control the routing of its own assigned traffic. In a Nash equilibrium, each user selfishly routes its traffic on those links that minimize its expected latency cost. The social cost of a Nash equilibrium is the expectation, over all random choices of the users, of the maximum, over all links, latency through a link.

We provide substantial evidence for the Fully Mixed Nash Equilibrium Conjecture, which states that the worst Nash equilibrium is the fully mixed Nash equilibrium, where each user chooses each link with positive probability. Specifically, we prove that the Fully Mixed Nash Equilibrium Conjecture is valid for pure Nash equilibria. Furthermore, we show, that under a certain condition, the social cost of any Nash equilibrium is within a factor of \( 2h(1+\varepsilon) \) of that of the fully mixed Nash equilibrium, where \( h \) is the factor by which the largest user traffic deviates from the average user traffic.

\textsuperscript{*}This work has been partially supported by the IST Program of the European Union under contract numbers IST-1999-14186 (ALCOM-FT) and IST-2001-33116 (FLAGS), by funds from the Joint Program of Scientific and Technological Collaboration between Greece and Cyprus, and by research funds at University of Cyprus.

E-mail addresses: gairing@upb.de (M. Gairing), luck@upb.de (T. Lücking), mavronic@ucy.ac.cy (M. Mavronicolas), bm@upb.de (B. Monien), spirakis@cti.gr (P. Spirakis).

0304-3975/ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2005.05.011
Considering pure Nash equilibria, we provide a PTAS to approximate the best social cost, we give an upper bound on the worst social cost and we show that it is \(NP\)-hard to approximate the worst social cost within a multiplicative factor better than \(2 - 2/(m + 1)\).

© 2005 Elsevier B.V. All rights reserved.

Keywords: Selfish routing; Extreme Nash equilibria

1. Introduction

1.1. Motivation and framework

A Nash equilibrium [22,23] represents a stable state of the play of a strategic game, in which each player holds an accurate opinion about the (expected) behavior of other players and acts rationally. An issue that arises naturally in this context concerns the computational complexity of Nash equilibria of any given strategic game. Due to the ultimate significance of Nash equilibrium as a prime solution concept in contemporary Game Theory [24], this issue has become a fundamental algorithmic problem that is being intensively studied in the Theory of Computing community today (see, e.g., [4,7,31]); in fact, it is arguably one of the few, most important algorithmic problems for which no general polynomial-time algorithms are known today (cf. [26]).

The problem of computing arbitrary Nash equilibria becomes even more challenging when one considers extreme Nash equilibria, ones that maximize or minimize a certain objective function. So, understanding the combinatorial structure of extreme Nash equilibria is a necessary prerequisite to either designing efficient algorithms to compute them or establishing corresponding hardness and thereby designing efficient approximation algorithms. In this work, we embark on a systematic study of the combinatorial structure and the computational complexity of extreme Nash equilibria; our study is carried out within the context of a simple selfish routing game, originally introduced in a pioneering work by Koutsoupias and Papadimitriou [16], that we describe next.

We assume a collection of \(n\) users, each employing a mixed strategy, which is a probability distribution over \(m\) parallel links, to control the shipping of its own assigned traffic. For each link, a capacity specifies the rate at which the link processes traffic. In a Nash equilibrium, each user selfishly routes its traffic on those links that minimize its expected latency cost, given the network congestion caused by the other users. A user’s support is the set of those links on which it may ship its traffic with non-zero probability. The social cost of a Nash equilibrium is the expectation, over all random choices of the users, of the maximum, over all links, latency through a link.

Our study distinguishes between pure Nash equilibria, where each user chooses exactly one link (with probability one), and mixed Nash equilibria, where the choices of each user are modeled by a probability distribution over links. We also distinguish in some cases between models of identical capacities, where all link capacities are equal, and of arbitrary capacities.
1.2. The fully mixed Nash equilibrium conjecture

In this work, we formulate and study a natural conjecture asserting that the fully mixed Nash equilibrium $F$ is the worst Nash equilibrium with respect to social cost. Formally, we conjecture:

**Conjecture 1.1 (Fully Mixed Nash Equilibrium Conjecture).** For any traffic vector $w$ such that the fully mixed Nash equilibrium $F$ exists, and for any Nash equilibrium $P$, $\text{SC}(w, P) \leq \text{SC}(w, F)$.

Clearly, the Fully Mixed Nash Equilibrium Conjecture is intuitive and natural: the fully mixed Nash equilibrium favors “collisions” between different users (since each user assigns its traffic with positive probability to every link); thus, this increased probability of “collisions” favors a corresponding increase to the (expected) maximum total traffic through a link, which is, precisely, the social cost. More importantly, the Fully Mixed Nash Equilibrium Conjecture is also significant since it precisely identifies the worst possible Nash equilibrium for the selfish routing game we consider; this will enable designers of Internet protocols not only to avoid choosing the worst-case Nash equilibrium, but also to calculate the worst-case loss to the system at any Nash equilibrium due to its deliberate lack of coordination, and to evaluate the Nash equilibrium of choice against the (provably) worst-case one.

1.3. Contribution and significance

Our study provides quite strong evidence in support of the Fully Mixed Nash Equilibrium Conjecture by either establishing or near establishing the conjecture in a number of interesting instances of the problem.

We start with the model of arbitrary capacities, where traffics are allowed to vary arbitrarily. There we prove that the Fully Mixed Nash Equilibrium Conjecture holds for pure Nash equilibria. We next turn to the case of identical capacities. Through a delicate probabilistic analysis, we establish that in the special case, the number of links is equal to the number of users and for a suitable large number of users, the social cost of any Nash equilibrium is less than $2h(1 + \epsilon)$ (for any $\epsilon > 0$) times the social cost of the fully mixed Nash equilibrium, where $h$ is the factor by which the largest user traffic deviates from the average user traffic. Our proof employs concepts and techniques from majorization theory [18] and stochastic orders [30], such as comparing two random variables according to their stochastic variability (cf. [28, Section 9.5]).

For pure Nash equilibria we show that it is $\mathcal{NP}$-hard to decide whether or not any given allocation of users to links can be transformed into a pure Nash equilibrium using at most $k$ selfish steps, even if the number of links is 2. Furthermore, we prove that there exists a polynomial-time approximation scheme (PTAS) to approximate the social cost of the best pure Nash equilibrium to any arbitrary accuracy. The proof involves an algorithm that transforms any pure strategy profile into a pure Nash equilibrium with at most the same social cost, using at most $n$ reassignments of users. We call this technique *Nashification*, and it may apply to other instances of the problem as well.
Still for pure Nash equilibria, we give a tight upper bound on the ratio between $SC(w, L)$ and $OPT(w)$ for any Nash equilibrium $L$. Then we show that it is $NP$-hard to approximate the worst-case Nash equilibrium with a ratio that is better than this upper bound. We close our section about pure Nash equilibria with a pseudopolynomial algorithm for computing the worst-case Nash equilibrium for any fixed number of links.

1.4. Related work and comparison

The selfish routing game considered in this paper was first introduced by Koutsoupias and Papadimitriou [16] as a vehicle for the study of the price of selfishness for routing over non-cooperative networks, subsequently studied in the work of Mavronicolas and Spirakis [19], where fully mixed Nash equilibria were introduced and analyzed. In both works, the aim had been to quantify the amount of performance loss in routing due to selfish behavior of the users. (Later studies of the selfish routing game from the same point of view, that of performance, include the works by Koutsoupias et al. [15] and by Czumaj and Vöcking [2].)

The closest to our work is the one by Fotakis et al. [7], which focuses on the combinatorial structure and the computational complexity of Nash equilibria for the selfish routing game we consider. The Fully Mixed Nash Equilibrium Conjecture formulated and systematically studied in this paper has been inspired by two results due to Fotakis et al. [7] that confirm or support the conjecture. First, Fotakis et al. [7, Theorem 4.2] establish the Fully Mixed Nash Equilibrium Conjecture for the model of identical capacities and assuming that $n = 2$. Second, Fotakis et al. [7, Theorem 4.3] establish that, for the model of arbitrary capacities, the social cost of any Nash equilibrium is no more than $49\sqrt{a}$ times the social cost of the (generalized) fully mixed Nash equilibrium.

The routing problem considered in this paper is equivalent to the multiprocessor scheduling problem. Here, pure Nash equilibria and Nashification translate to local optima and sequences of local improvements. A schedule is said to be jump optimal if no job on a processor with maximum load can improve by moving to another processor [29]. Obviously, the set of pure Nash equilibria is a subset of the set of jump optimal schedules. Moreover, in the model of identical processors every jump optimal schedule can be transformed into a pure Nash equilibrium without altering the makespan. Thus, for this model the strict upper bound $2 - 2/(m + 1)$ on the ratio between best and worst makespan of jump optimal schedules [6,29] also holds for pure Nash equilibria.

Algorithms for computing a jump optimal schedule from any given schedule have been proposed in [1,6,29]. The fastest algorithm is given by Schuurman and Vredeveld [29]. It always moves the job with maximum weight from a makespan processor to a processor with minimum load, using $O(n)$ moves. However, in all algorithms the resulting jump optimal schedule is not necessarily a Nash equilibrium.

1.5. Road map

The rest of this paper is organized as follows. Section 2 presents some preliminaries. Stochastic orders are treated in Section 3. Pure Nash equilibria are contrasted to the fully mixed Nash equilibrium in Section 4. Worst mixed Nash equilibria are contrasted to the
fully mixed Nash equilibrium in Section 5. Sections 6 and 7 consider best and worst pure Nash equilibria, respectively. We conclude, in Section 8, with a discussion of our results and some open problems.

2. Framework

Most of our definitions are patterned after those in [19, Section 2] and [7, Section 2], which, in turn, were based on those in [16, Sections 1 and 2].

2.1. Mathematical preliminaries and notation

For any integer \( m \geq 1 \), denote \([m] = \{1, \ldots, m\}\). Denote \( \Gamma \) the Gamma function; that is, for any natural number \( N \), \( \Gamma(N + 1) = N! \), while for any arbitrary real number \( x > 0 \), \( \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \). The Gamma function is invertible; both \( \Gamma \) and its inverse \( \Gamma^{-1} \) are increasing. It is well known that \( \Gamma^{-1}(N) = (\log N / \log \log N)(1 + o(1)) \) (see, e.g., [10]).

For our purposes, we shall use the fact that for any pair of an arbitrary real number \( z \) and an arbitrary natural number \( N, (z/e)^2 = N \) if and only if \( z = \Gamma^{-1}(N) + \Theta(1) \). For an event \( E \) in a sample space, denote \( \Pr(E) \) the probability of event \( E \) happening.

For a random variable \( X \), denote \( \mathcal{E}(X) \) the expectation of \( X \). In the balls-and-bins problem, \( m \) balls are thrown into \( m \) bins uniformly at random. (See [14] for a classical introduction to this problem.) It is known that the expected maximum number of balls thrown over a bin equals the quantity \( R(m) = \Gamma^{-1}(m) - \frac{3}{2} + o(1) \) [10].

In the paper, we make use of the following inequality, which holds due to Hoeffding.

**Theorem 2.1 (McDiarmid [20, Theorem 2.3]).** Let \( X_1, X_2, \ldots, X_n \) be independent random variables with \( 0 \leq X_k \leq 1 \) for each \( k \). Let \( S_n = \sum X_k \) and \( \mu = \mathcal{E}(S_n) \). Then, for any \( \beta > 0 \),

\[
\Pr(S_n \geq (1 + \beta)\mu) \leq e^{-(1+\beta)\beta(1+\beta)\mu}.
\]

Note that if \( 0 \leq X_k \leq \kappa \) for all \( k \in [n] \) and for some constant \( \kappa > 0 \), then for any \( \beta > 0 \),

\[
\Pr(S_n \geq (1 + \beta)\mu) \leq e^{-(1+\beta)\beta(1+\beta)\mu} \frac{\mu^\kappa}{\kappa!}.
\]

2.2. General

We consider a network consisting of a set of \( m \) parallel links \( 1, 2, \ldots, m \) from a source node to a destination node. Each of \( n \) network users \( 1, 2, \ldots, n \), or users for short, wishes to route a particular amount of traffic along a (non-fixed) link from source to destination. Denote \( w_i \) the traffic of user \( i \in [n] \). Define the \( n \times 1 \) traffic vector \( w \) in the natural way. Assume throughout that \( m > 1 \) and \( n > 1 \). Assume also, without loss of generality, that \( w_1 \geq w_2 \geq \cdots \geq w_n \). For a traffic vector \( w \), denote \( W = \sum_1^n w_i \). Define \( h \) as the factor by which the largest user traffic deviates from the average user traffic, thus, \( h = w_1 / W \).

A pure strategy for user \( i \in [n] \) is some specific link. A mixed strategy for user \( i \in [n] \) is a probability distribution over pure strategies; thus, a mixed strategy is a probability
distribution over the set of links. The support of the mixed strategy for user $i \in [n]$, denoted $\text{support}(i)$, is the set of those pure strategies (links) to which $i$ assigns positive probability.

A pure strategy profile is represented by an $n$-tuple $(\ell_1, \ell_2, \ldots, \ell_n) \in [m]^n$; a mixed strategy profile is represented by an $n \times m$ probability matrix $P$ of $nm$ probabilities $p^j_i$, $i \in [n]$ and $j \in [m]$, where $p^j_i$ is the probability that user $i$ chooses link $j$. For a probability matrix $P$, define indicator variables $I^i_\ell \in \{0, 1\}$, $i \in [n]$ and $\ell \in [m]$, such that $I^i_\ell = 1$ if and only if $p^j_i > 0$. Thus, the support of the mixed strategy for user $i \in [n]$ is the set $\{\ell \in [m] | I^i_\ell = 1\}$.

For each link $\ell \in [m]$, define the view of link $\ell$, denoted $\text{view}(\ell)$, as the set of users $i \in [n]$ that potentially assign their traffic to link $\ell$; so, $\text{view}(\ell) = \{i \in [n] | I^i_\ell = 1\}$. For each link $\ell \in [m]$, denote $V^\ell = |\text{view}(\ell)|$. A mixed strategy profile $P$ is fully mixed [19, Section 2.2] if for all users $i \in [n]$ and links $j \in [m]$, $I^i_\ell = 1$.

### 2.3. System, models and cost measures

Denote $c^\ell > 0$ the capacity of link $\ell \in [m]$, representing the rate at which the link processes traffic. So, the latency for traffic $w$ through link $\ell$ equals $w/c^\ell$. In the model of identical capacities, all link capacities are equal to 1; link capacities may vary arbitrarily in the model of arbitrary capacities. For a pure strategy profile $(\ell_1, \ell_2, \ldots, \ell_n)$, the latency cost for user $i$, denoted $\lambda_i$, is $(\sum_{k : I^i_k = 1} w_k)/c^\ell$; that is, the latency cost for user $i$ is the latency of the link it chooses. For a mixed strategy profile $P$, denote $\delta^\ell$ the actual traffic on link $\ell \in [m]$; so, $\delta^\ell$ is a random variable for each link $\ell \in [m]$, denote $\theta^\ell$ the expected traffic on link $\ell \in [m]$; thus, $\theta^\ell = \mathbb{E}(\delta^\ell) = \sum_{i=1}^n p^j_i w_i$. Given $P$, define the $m \times 1$ expected traffic vector $\Theta$ induced by $P$ in the natural way. Given $P$, denote $A^\ell$ the expected latency on link $\ell \in [m]$; clearly, $A^\ell = \theta^\ell/c^\ell$. Define the $m \times 1$ expected latency vector $A$ in the natural way. For a mixed strategy profile $P$, the expected latency cost for user $i \in [n]$ on link $\ell \in [m]$, denoted $\lambda^i_\ell$, is the expectation, over all random choices of the remaining users, of the latency cost for user $i$ had its traffic been assigned to link $\ell$; thus,

$$\lambda^i_\ell = \frac{w_i + \sum_{k=1, k \neq i}^n p^j_k w_k}{c^\ell} = \frac{(1 - p^j_i) w_i + \theta^\ell}{c^\ell}.$$

For each user $i \in [n]$, the minimum expected latency cost, denoted $\lambda_i$, is the minimum, over all links $\ell \in [m]$, of the expected latency cost for user $i$ on link $\ell$; thus, $\lambda_i = \min_{\ell \in [m]} \lambda^i_\ell$. For a probability matrix $P$, define the $n \times 1$ minimum expected latency cost vector $\lambda$ induced by $P$ in the natural way.

Associated with a traffic vector $w$ and a mixed strategy profile $P$ is the social cost [16, Section 2], denoted $SC(w, P)$, which is the expectation, over all random choices of the

---

1 An earlier treatment of fully mixed strategies in the context of bimatrix games has been found in [27], called there completely mixed strategies. See also [21] for a subsequent treatment in the context of strategically zero-sum games. Datta [3] studied recently some universality properties of fully mixed Nash equilibria (calling them totally mixed).
users, of the maximum (over all links) latency of traffic through a link; thus,

\[ SC(w, P) = E\left( \max_{\ell \in [m]} \sum_{k: \ell_k = \ell} \frac{w_k}{c^\ell} \right) = \sum_{(\ell_1, \ell_2, \ldots, \ell_n) \in [m]^n} \left( \prod_{k=1}^n p_k^{\ell_k} \cdot \max_{\ell \in [m]} \sum_{k: \ell_k = \ell} \frac{w_k}{c^\ell} \right). \]

Note that \( SC(w, P) \) reduces to the maximum latency through a link in the case of pure strategies. On the other hand, the social optimum \[16, Section 2\] associated with a traffic vector \( w \), denoted \( \text{OPT}(w) \), is the least possible maximum (over all links) latency of traffic through a link; thus,

\[ \text{OPT}(w) = \min_{(\ell_1, \ell_2, \ldots, \ell_n) \in [m]^n} \max_{\ell \in [m]} \sum_{k: \ell_k = \ell} \frac{w_k}{c^\ell}. \]

### 2.4. Nash equilibria

We are interested in a special class of mixed strategies called Nash equilibria \[22,23\] that we describe below. Say that a user \( i \in [n] \) is satisfied for the probability matrix \( P \) if for all links \( \ell \in [m] \), \( \lambda_i^\ell = \lambda_i \) if \( I_i^\ell = 1 \), and \( \lambda_i^\ell > \lambda_i \) if \( I_i^\ell = 0 \); thus, a satisfied user has no incentive to unilaterally deviate from its mixed strategy. A user \( i \in [n] \) is unsatisfied for the probability matrix \( P \) if \( i \) is not satisfied for the probability matrix \( P \). The probability matrix \( P \) is a Nash equilibrium \[16, Section 2\] if for all users \( i \in [n] \) and links \( \ell \in [m] \), \( \lambda_i^\ell = \lambda_i \) if \( I_i^\ell = 1 \), and \( \lambda_i^\ell > \lambda_i \) if \( I_i^\ell = 0 \). Thus, each user assigns its traffic with positive probability only on links (possibly more than one of them) for which its expected latency cost is minimized. The fully mixed Nash equilibrium \[19\], denoted \( F \), is a Nash equilibrium that is a fully mixed strategy. Mavronicolas and Spirakis \[19, Lemma 15\] show that all links are equiprobable in a fully mixed Nash equilibrium, which is unique (for the model of identical capacities).

Fix any traffic vector \( w \). The worst Nash equilibrium is the Nash equilibrium \( P \) that maximizes \( SC(w, P) \); the best Nash equilibrium is the Nash equilibrium that minimizes \( SC(w, P) \). The worst social cost, denoted \( \text{WC}(w) \), is the social cost of the worst Nash equilibrium; correspondingly, the best social cost, denoted \( \text{BC}(w) \), is the social cost of the best Nash equilibrium.

Fotakis et al. \[7, Theorem 1\] consider starting from any arbitrary pure strategy profile and following a particular sequence of selfish steps, where in a selfish step, exactly one unsatisfied user is allowed to change its pure strategy. A selfish step is a greedy selfish step if the unsatisfied user chooses its best link. A (greedy) selfish step does not increase the social cost of the initial pure strategy profile. Fotakis et al. \[7, Theorem 1\] show that this sequence of selfish steps eventually converges to a Nash equilibrium, which proves its existence; however, it may take a large number of steps. It follows that if the initial pure strategy profile has minimum social cost, then the resulting (pure) Nash equilibrium will have minimum social cost as well. This implies that there exists a pure Nash equilibrium with minimum social cost. Thus, we have \( \text{BC}(w) = \text{OPT}(w) \).
2.5. Algorithmic problems

We list a few algorithmic problems related to Nash equilibria that will be considered in this work. The definitions are given in the style of Garey and Johnson [9]. A problem instance is a tuple \((n, m, w, c)\), where \(n\) is the number of users, \(m\) is the number of links, \(w = (w_i)\) is a vector of \(n\) user traffics and \(c = (c^j)\) is a vector of \(m\) link capacities.

\[\Pi_1: \text{NASH EQUILIBRIUM SUPPORTS}\]

INSTANCE: A problem instance \((n, m, w, c)\).

OUTPUT: Indicator variables \(I^i_j \in \{0, 1\}\), where \(i \in [n]\) and \(j \in [m]\), that support a Nash equilibrium for the system of the users and the links.

Fotakis et al. [7, Theorem 2] establish that \text{NASH EQUILIBRIUM SUPPORTS} is in \(P\) when restricted to pure equilibria. We continue with two complementary to each other optimization problems (with respect to social cost).

\[\Pi_2: \text{BEST NASH EQUILIBRIUM SUPPORTS}\]

INSTANCE: A problem instance \((n, m, w, c)\).

OUTPUT: Indicator variables \(I^i_j \in \{0, 1\}\), where \(i \in [n]\) and \(j \in [m]\), that support the best Nash equilibrium for the system of the users and the links.

\[\Pi_3: \text{WORST NASH EQUILIBRIUM SUPPORTS}\]

INSTANCE: A problem instance \((n, m, w, c)\).

OUTPUT: Indicator variables \(I^i_j \in \{0, 1\}\), where \(i \in [n]\) and \(j \in [m]\), that support the worst Nash equilibrium for the system of the users and the links.

Fotakis et al. [7, Theorems 3 and 4] establish that both \text{BEST NASH EQUILIBRIUM SUPPORTS} and \text{WORST NASH EQUILIBRIUM SUPPORTS} are \(\mathcal{NP}\)-hard. Since both problems can be formulated as an integer program, it follows that they are \(\mathcal{NP}\)-complete.

\[\Pi_4: \text{NASH EQUILIBRIUM SOCIAL COST}\]

INSTANCE: A problem instance \((n, m, w, c)\); a Nash equilibrium \(P\) for the system of the users and the links.

OUTPUT: The social cost of the Nash equilibrium \(P\).

Fotakis et al. [7, Theorem 8] establish that \text{NASH EQUILIBRIUM SOCIAL COST} is \(\#P\)-complete. Furthermore, Fotakis et al. [7, Theorem 9] show that there exists a fully polynomial, randomized approximation scheme for \text{NASH EQUILIBRIUM SOCIAL COST}.

The following two problems, inspired by \text{NASH EQUILIBRIUM SOCIAL COST}, are introduced for the first time in this work.

\[\Pi_5: \text{WORST NASH EQUILIBRIUM SOCIAL COST}\]

INSTANCE: A problem instance \((n, m, w, c)\).

OUTPUT: The worst social cost \(\text{WSC}(w)\).

\[\Pi_6: \text{BEST NASH EQUILIBRIUM SOCIAL COST}\]

INSTANCE: A problem instance \((n, m, w, c)\).

OUTPUT: The best social cost \(\text{BSC}(w)\).
\( \Pi_7: k\)-NASHIFY

INSTANCE: A problem instance \((n, m, w, c)\); a pure strategy profile \(L\) for the system of the users and the links.

QUESTION: Is there a sequence of at most \(k\) selfish steps that transform \(L\) to a (pure) Nash equilibrium?

The following problem is a variant of \(k\)-NASHIFY in which \(k\) is part of the input.

\( \Pi_8: \) NASHIFY

INSTANCE: A problem instance \((n, m, w, c)\); a pure strategy profile \(L\) for the system of the users and the links; an integer \(k > 0\).

QUESTION: Is there a sequence of at most \(k\) selfish steps that transform \(L\) to a (pure) Nash equilibrium?

In our hardness and completeness proofs, we will employ the following \(NP\)-complete problems [13]:

\( \Pi_9: \) BIN PACKING

INSTANCE: A finite set \(U\) of items, a size \(s(u) \in \mathbb{N}\) for each \(u \in U\), a positive integer bin capacity \(B\), and a positive integer \(K\).

QUESTION: Is there a partition of \(U\) into disjoint sets \(U_1, \ldots, U_K\) such that for each set \(U_i\), \(1 \leq i \leq K\), \(\sum_{u \in U_i} s(u) \leq B\)?

\( \Pi_{10}: \) PARTITION

INSTANCE: A finite set \(U\) and a size \(s(u) \in \mathbb{N}\) for each element \(u \in U\).

QUESTION: Is there a subset \(U' \subseteq U\) such that \(\sum_{u \in U'} s(u) = \sum_{u \in U \setminus U'} s(u)\)?

We note that BIN PACKING is strongly \(NP\)-complete [8,25].

3. Stochastic order relations

In this section, we treat stochastic order relations; we establish a certain stochastic order relation for the expected maxima of certain sums of Bernoulli random variables. We will show that in the balls-and-bins game \((m\) balls are thrown at random into \(m\) bins), if the sum of the ball weights is the same, the expected maximum load over all bins is larger when the balls have different weight in comparison to all balls having the same weight. This will be used in Section 5 to prove an upper bound on the social cost of a worst mixed Nash equilibrium.

Recall that a function \(f : \mathbb{R} \rightarrow \mathbb{R}\) is convex if for all numbers \(\lambda\) such that \(0 < \lambda < 1\), \(f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)\). We proceed to describe a stochastic order relation between two random variables.

**Definition 3.1.** For any pair of arbitrary random variables \(X\) and \(Y\), say that \(X\) is stochastically more variable than \(Y\) if for all increasing and convex functions \(f : \mathbb{R} \rightarrow \mathbb{R}\), \(\mathbb{E}(f(X)) \geq \mathbb{E}(f(Y))\).

\(^2\) A problem is strongly \(NP\)-complete if it remains \(NP\)-complete even if any instance of length \(n\) is restricted to contain integers of size polynomial in \(n\). So, strongly \(NP\)-complete problems admit no pseudopolynomial-time algorithms unless \(P = NP\).
Call \textit{stochastically more variability} the corresponding stochastic order relation on the set of random variables. (See [28, Section 9.5] for a more complete treatment of the notion of stochastically more variable and [18,30] for more on majorization theory and stochastic orders.) The following lemma [28, Proposition 9.5.1] provides an alternative, analytic characterization of stochastically more variability.

\textbf{Lemma 3.1.} Consider any pair of non-negative random variables \(X\) and \(\hat{X}\). Then, \(X\) is stochastically more variable than \(\hat{X}\) if and only if for all numbers \(x \geq 0\), \(\int_{x=0}^{\infty} \Pr(X > x) \, dx \geq \int_{x=0}^{\infty} \Pr(\hat{X} > x) \, dx\).

Consider now a setting of the balls-and-bins problem where \(n\) balls \(1, \ldots, n\) with \textit{traffic} \(w_1, \ldots, w_n\) are allocated into \(m\) bins \(1, \ldots, m\) uniformly at random. So, for each pair of a ball \(i \in [n]\) and a link \(j \in [m]\), define Bernoulli random variables \(Y_j^i = w_i \) with probability \(1/m\) and \(0\) with probability \(1 - 1/m\), and \(\tilde{Y}_j^i = W/n \) with probability \(1/m\) and \(0\) with probability \(1 - 1/m\). For each link \(j \in [m]\), define the random variables \(\delta_j^i = \sum_{i \in [n]} Y_j^i\) and \(\tilde{\delta}_j^i = \sum_{i \in [n]} \tilde{Y}_j^i\); thus, each of \(\delta_j^i\) and \(\tilde{\delta}_j^i\), \(j \in [m]\), is a sum of Bernoulli random variables; denote \(\theta_j^i = E(\delta_j^i)\) and \(\tilde{\theta}_j^i = E(\tilde{\delta}_j^i)\) the expectations of \(\delta_j^i\) and \(\tilde{\delta}_j^i\), respectively. Note that

\[
\theta_j^i = E\left( \sum_{i \in [n]} Y_j^i \right) = \sum_{i \in [n]} E(Y_j^i) = \sum_{i \in [n]} \left( w_i \frac{1}{m} + 0 \left( 1 - \frac{1}{m} \right) \right) = \sum_{i \in [n]} \frac{w_i}{m} = \frac{W}{m},
\]

while

\[
\tilde{\theta}_j^i = E(\tilde{\delta}_j^i) = E\left( \sum_{i \in [n]} \tilde{Y}_j^i \right) = \sum_{i \in [n]} E(Y_j^i) = \sum_{i \in [n]} \left( \frac{W}{n} \frac{1}{m} + 0 \left( 1 - \frac{1}{m} \right) \right) = \frac{W}{n} \frac{1}{m} = \frac{W}{m}.
\]

So, \(\theta_j^i = \tilde{\theta}_j^i\) for each bin \(j \in [m]\).

For two numbers \(x, y \in \mathbb{R}^+\) define

\[
[x - y] = \begin{cases} 
  x - y & \text{if } x > y, \\
  0 & \text{else}.
\end{cases}
\]

We can then show the following preliminary lemma:

\textbf{Lemma 3.2.} Let \(b_i \in \mathbb{R}^+\) for \(i \in [n]\) and let \(d = (1/n) \sum_{i=1}^{n} b_i\). Then for all \(x \geq 0\)

\[
\sum_{i=1}^{n} [b_i - x] \geq n \cdot [d - x].
\]
Proof. Without loss of generality, assume that $b_1 \leq b_2 \leq \cdots \leq b_n$. The claim is true if $x > d$. If $x \leq b_1$, then $x \leq d$ and

$$\sum_{i=1}^{n} [b_i - x] = \sum_{i=1}^{n} (b_i - x) = n \cdot (d - x).$$

Now let $b_j < x \leq b_{j+1}$ and $d > x$. It follows that

$$\sum_{i=1}^{n} [b_i - x] = \sum_{i=j+1}^{n} (b_i - x) = \sum_{i=j+1}^{n} b_i - n \cdot x + j \cdot x$$

$$\geq \sum_{i=j+1}^{n} b_i - n \cdot x + \sum_{i=1}^{j} b_i = \sum_{i=1}^{n} b_i - n \cdot x$$

$$= n \cdot (d - x). \quad \square$$

We finally prove:

Lemma 3.3 (Stochastically More Variability Lemma). For any traffic vector $w$, $\max \{\delta^1, \ldots, \delta^m\}$ is stochastically more variable than $\max \{\tilde{\delta}^1, \ldots, \tilde{\delta}^m\}$.

Proof. Define the discrete random variables $X = \max \{\delta^1, \ldots, \delta^m\}$ and $\tilde{X} = \max \{\tilde{\delta}^1, \ldots, \tilde{\delta}^m\}$. We then have to show that for all $x \geq 0$,

$$\int_{x=0}^{\infty} \Pr(X > x) \, dx \geq \int_{x=0}^{\infty} \Pr(\tilde{X} > x) \, dx.$$

Let $S_k$ be the collection of all pure strategy profiles, where the maximum number of traffics on any link $j \in [m]$ is exactly $k$. If $i \neq j$, then $S_i \cap S_j = \emptyset$. Furthermore

$$\bigcup_{i=\lceil n/m \rceil}^{n} S_i = [m]^n.$$  

For any pure strategy profile $L \in S_k$, define $\text{Link}(L)$ to be the smallest index of a link, holding $k$ traffics. Furthermore, for any pure strategy profile $L$, let $I(L)$ be the collection of users that are assigned to $\text{Link}(L)$. Every set of $k$ traffics is equal to some $I(L)$, $L \in S_k$ with the same probability, say $p_k$. Define the actual traffic on $\text{Link}(L)$ as

$$b(L) = \sum_{i \in I(L)} w_i.$$  

If all traffics are identical the actual traffic on $\text{Link}(L)$ for a pure strategy profile $L \in S_k$ is simply $\tilde{b}(L) = k \cdot W/n$.

Every pure strategy profile $L \in [m]^n$ occurs with the same probability $1/m^n$ and defines together with $b(L)$ a discrete random variable $Z$. $Z$ is a discrete random variable that can take every possible value $b(L)$, $L \in [m]^n$.

It is easy to see that $X$ is stochastically more variable than $Z$, since for any pure strategy profile $L$, $Z$ refers to the actual traffic on $\text{Link}(L)$, whereas $X$ refers to the maximum actual traffic over all links.
We will complete our proof by showing that $Z$ is stochastically more variable than $\tilde{X}$.

Since $Z$ and $\tilde{X}$ are discrete random variables,

$$\int_{x=\varepsilon}^{\infty} \Pr(Z > x) \, dx = \sum_{k=[n/m]}^{n} (p_k \cdot A_k), \quad \text{where } A_k = \sum_{L \in S_k} [b(L) - \varepsilon]$$

and

$$\int_{x=\varepsilon}^{\infty} \Pr(\tilde{X} > x) \, dx = \sum_{k=[n/m]}^{n} (p_k \cdot \tilde{A}_k), \quad \text{where } \tilde{A}_k = |S_k| \cdot \left[ k \cdot \frac{W}{n} - \varepsilon \right].$$

Since for a fixed $k$ each traffic contributes with the same probability to $b(L)$,

$$\sum_{L \in S_k} b(L) = |S_k| \cdot k \cdot \frac{W}{n}.$$ 

It follows from Lemma 3.2 that $A_k \geq \tilde{A}_k$ for each $k$. Therefore $Z$ is stochastically more variable than $\tilde{X}$, which completes the proof of the lemma. □

By definition of stochastically more variability, Lemma 3.3 immediately implies:

**Corollary 3.4.** For any traffic vector $w$,

$$\mathcal{E}(\max\{\delta^1, \ldots, \delta^m\}) \geq \mathcal{E}(\max\{\tilde{\delta}^1, \ldots, \tilde{\delta}^m\}).$$

In the balls-and-bins game in which $m$ balls are thrown uniformly at random into $m$ bins, Corollary 3.4 shows that if the sum of the ball weights is the same, the expected maximum load over all bins is larger when the balls have different weights in comparison to all balls having the same weight.

### 4. Pure versus fully mixed Nash equilibria

In this section, we establish the Fully Mixed Nash Equilibrium Conjecture for the case of pure Nash equilibria. This result holds also for the model of arbitrary capacities.

We show that the minimum expected latency cost of a user in any (mixed) Nash equilibrium is at most its minimum expected latency cost in the fully mixed Nash equilibrium. Afterwards we prove that this implies validity of the Fully Mixed Nash Equilibrium Conjecture for pure Nash equilibria.

We start by proving:

**Lemma 4.1.** Fix any traffic vector $w$, mixed Nash equilibrium $P$ and user $i$. Then, $\lambda_i(w, P) \leq \lambda_i(w, F)$.

**Proof.** Let $P = (p_k^j)$, $F = (f_k^j)$ for $k \in [n]$ and $j \in [m]$. Then

$$\sum_{j \in [m]} \left( \sum_{k \in [n], k \neq i} p_k^j w_k \right) = \sum_{k \in [n], k \neq i} w_k \left( \sum_{j \in [m]} p_k^j \right) = \sum_{k \in [n], k \neq i} w_k$$
and
\[ \sum_{j \in [m]} \left( \sum_{k \in [n], k \neq i} f^j_k w_k \right) = \sum_{k \in [n], k \neq i} w_k \left( \sum_{j \in [m]} f^j_k \right) = \sum_{k \in [n], k \neq i} w_k. \]

It follows that
\[ \sum_{j \in [m]} \left( \sum_{k \in [n], k \neq i} p^j_k w_k \right) = \sum_{j \in [m]} \left( \sum_{k \in [n], k \neq i} f^j_k w_k \right), \]
and therefore there exists some link \( j_0 \in [m] \) such that
\[ \sum_{k \in [n], k \neq i} p^j_0 w_k \leq \sum_{k \in [n], k \neq i} f^j_0 w_k. \]

Then,
\[ \lambda_i(w, P) \leq \lambda_i^j(w, P) \quad \text{(since } \lambda_i \text{ is the minimum of all } \lambda_i^j, j \in [n]) \]
\[ = w_i + \sum_{k \in [n], k \neq i} p^j_k w_k \]
\[ \leq w_i + \sum_{k \in [n], k \neq i} f^j_k w_k \]
\[ = \lambda_i^j(w, F) \]
\[ = \lambda_i(w, F) \quad \text{(since } f^j_i > 0 \text{ and } F \text{ is a Nash equilibrium).} \]

The following theorem shows that the Fully Mixed Nash Equilibrium Conjecture is valid for pure Nash equilibria.

**Theorem 4.2.** Fix any traffic vector \( w \) and pure Nash equilibrium \( L \). Then, \( SC(w, L) \leq SC(w, F) \).

**Proof.** For each user \( i \in [n] \), \( \lambda_i(w, P) \) is the minimum, over all links \( j \in [m] \), of the expected latency cost for user \( i \) on link \( j \), and \( SC(w, P) \) is the expectation of the maximum (over all links) latency of traffic through a link. This implies that \( \lambda_i(w, P) \leq SC(w, P) \) for every mixed Nash equilibrium \( P \). Hence,
\[ \lambda_i(w, P) \leq \lambda_i(w, F) \quad \text{(by Lemma 4.1)} \]
\[ \leq SC(w, F) \quad \text{(as shown above).} \]

The claim follows now since \( SC(w, L) = \max_{i \in [n]} \lambda_i(w, L) \) holds for every pure Nash equilibrium \( L \).

**5. Worst mixed Nash equilibria**

In this section we show that if \( n = m \) and \( m \) is suitably large then the social cost of any Nash equilibrium is at most \( 2h(1 + \varepsilon) \) times the social cost of the fully mixed Nash equilibrium. Recall, that \( h = \frac{w_{1n}}{w} = \frac{w_{1m}}{w} \).
Theorem 5.1. Consider the model of identical capacities. Let \( n = m \), \( m \) suitably large. Then, for any traffic vector \( w \) and Nash equilibrium \( P \), \( SC(w, P) < 2h(1 + \varepsilon) SC(w, F) \), for any \( \varepsilon > 0 \).

Proof. Fix any traffic vector \( w \) and Nash equilibrium \( P \). We start by showing a simple technical fact.

Claim 1. Fix any pair of a link \( \ell \in [m] \) and a user \( i \in \text{view}(\ell) \). Then,
\[
\theta^\ell_i w_i \geq \frac{1}{af^i} - \frac{W}{m}.
\]

Proof. Clearly,
\[
\sum_{j \in [m]} \theta^j = \sum_{j \in [m]} \left( \sum_{i \in [n]} p^j_i w_i \right) = \sum_{i \in [n]} \left( \sum_{j \in [m]} p^j_i w_i \right) = \sum_{i \in [n]} \left( w_i \sum_{j \in [m]} p^j_i \right) = \sum_{i \in [n]} w_i = W.
\]
This implies that there exists some link \( \ell' \in [m] \) such that \( \theta^\ell_i w_i \leq \frac{W}{m} \). Note that by definition of social cost, \( \lambda^\ell_i = (1 - p^\ell_i) w_i + \theta^\ell_i \). It follows that \( \lambda^\ell_i \leq w_i + \theta^\ell_i \). On the other hand, \( \lambda^\ell_i = (1 - p^\ell_i) w_i + \theta^\ell_i \).

Since \( i \in \text{view}(\ell) \), we have, by definition of Nash equilibria, that \( \lambda^\ell_i \leq \lambda_i^\ell' \) (with equality holding when \( i \in \text{view}(\ell') \)). It follows that \( (1 - p^\ell_i) w_i + \theta^\ell_i \leq w_i + \theta^\ell_i \), or that \( p^\ell_i w_i \geq \theta^\ell - \frac{W}{m} \), as needed. \( \square \)

As an immediate consequence of Claim 1, we obtain:

Corollary 5.2. Fix any link \( \ell \in [m] \). Then, \( \theta^\ell \leq \frac{V^\ell}{V^\ell - 1} W/m \).

Proof. Clearly, by Claim 1,
\[
\theta^\ell = \sum_{i \in [n]} p^\ell_i w_i = \sum_{i \in \text{view}(\ell)} p^\ell_i w_i \geq \sum_{i \in \text{view}(\ell)} \left( \theta^\ell - \frac{W}{m} \right) = V^\ell \left( \theta^\ell - \frac{W}{m} \right),
\]
or, by rearrangement of terms, \( \theta^\ell \leq \frac{V^\ell}{V^\ell - 1} W/m \), as needed. \( \square \)

Since \( V^\ell \geq 2 \), \( V^\ell/(V^\ell - 1) \leq 2 \). Thus, by Corollary 5.2:

Lemma 5.3. Fix any link \( \ell \in [m] \) with \( V^\ell \geq 2 \). Then, \( \theta^\ell \leq 2 W/m \).

We now prove a complementary lemma. Fix any link \( \ell \in [m] \) with \( V^\ell = 1 \). Let \( \text{view}(\ell) = \{i\} \). Then \( \theta^\ell \leq w_i \leq \max_i w_i \leq \text{OPT}(w) \leq SC(w, F) \). Thus:

Lemma 5.4. Fix any link \( \ell \in [m] \) with \( V^\ell = 1 \). Then, \( \theta^\ell \leq SC(w, F) \).

Use \( w \) to define the vector \( \tilde{w} \) with all entries equal to \( W/n \). By definition of social cost, \( SC(\tilde{w}, F) \) is the load \( W/m \) of each ball times the expected maximum number of balls thrown
uniformly at random into \( m \) bins. Since \( n = m \), we can state \( \text{SC}(\tilde{w}, F) = R(m) \cdot W/m \), or \( W/m = \text{SC}(\tilde{w}, F)/R(m) \). Fix now any link \( j \in [n] \) with \( V^j \geq 2 \). Then,

\[
0^j \leq \frac{2W}{m} \quad \text{(by Lemma 5.3)}
\]

\[
= \frac{2w_1}{h}
\]

Furthermore,

\[
\text{SC}(w, F) \geq \text{SC}(\tilde{w}, F) \quad \text{(by Corollary 3.4)}
\]

\[
= R(m) \frac{W}{m}
\]

\[
= R(m) \frac{w_1}{h} \quad \text{(by Definition of } h\text{)}.
\]

Let \( r \geq 2, r \in \mathbb{N} \). Then, for any constant \( \varepsilon > 0 \), arbitrarily close to 0,

\[
\Pr(\delta^j > rh(1 + \varepsilon) \text{SC}(w, F)) \leq \Pr(\delta^j > r(1 + \varepsilon) R(m)w_1) \quad \text{(since } \text{SC}(w, F) \geq R(m) \frac{W}{m} = R(m) \frac{w_1}{h} \text{)}.
\]

From Theorem 2.1 it follows that for any \( \beta > 0 \),

\[
\Pr(\delta^j \geq (1 + \beta) \mathcal{E}(\delta^j)) \leq e^{-\frac{(1 + \beta)(1 + \beta) \mathcal{E}(\delta^j)}{w_1}} = \frac{e^{\beta \mathcal{E}(\delta^j)/w_1}}{(1 + \beta)^{(1 + \beta) \mathcal{E}(\delta^j)/w_1}} < \left( \frac{e}{1 + \beta} \right)^{(1 + \beta) \mathcal{E}(\delta^j)/w_1}.
\]

With \( (1 + \beta) = r(1 + \varepsilon) R(m) \frac{w_1}{h} \) and since \( \mathcal{E}(\delta^j) \leq 2 \frac{w_1}{R(m)} \leq 2 w_1 \leq r w_1 \) we get:

\[
\Pr(\delta^j > rh(1 + \varepsilon) \text{SC}(w, F)) \leq \Pr(\delta^j > r(1 + \varepsilon) R(m)w_1)
\]

\[
< \left( \frac{e \cdot \mathcal{E}(\delta^j)}{r(1 + \varepsilon) R(m) w_1} \right)^{\frac{r(1 + \varepsilon) R(m) w_1}{w_1}} \leq \left( \frac{e}{(1 + \varepsilon) R(m)} \right)^{(1 + \varepsilon) R(m)} \leq \left( \frac{e}{(1 + \varepsilon) R(m)} \right)^{(1 + \varepsilon) R(m)}.
\]

Define now \( \alpha > 0 \) so that \( \alpha/e^\alpha = m \). Then, clearly, \( \alpha = \Gamma^{-1}(m) + \Theta(1) \). Note that

\[
(1 + \varepsilon) R(m) = (1 + \varepsilon) \Gamma^{-1}(m) - (1 + \varepsilon) \frac{2}{3} + o(1) \quad \text{(by definition of } R(m))
\]

\[
= (1 + \varepsilon) \Gamma^{-1}(m) + \Theta(1)
\]

\[
> \alpha \quad \text{(for suitably large } m, \text{ since } \varepsilon > 0).
\]
Since \((x/e)^x\) is an increasing function of \(x\), this implies that
\[
\left(\frac{(1+\varepsilon)R(m)}{e}\right)^{(1+\varepsilon)R(m)} > \left(\frac{x}{e}\right)^x = m.
\]
This implies that
\[
\left(\left(\frac{e}{(1+\varepsilon)R(m)}\right)^{(1+\varepsilon)R(m)}\right)^\varepsilon < \frac{1}{m^\varepsilon}.
\]
It follows that
\[
\mathbb{Pr}(\delta^j > rh(1 + \varepsilon) SC(w, F)) < \frac{1}{m^\varepsilon}.
\]
Hence
\[
\mathbb{Pr}
\left(\max_{\ell \in [m] \mid |V^\ell| \geq 2} \delta^\ell > rh(1 + \varepsilon) SC(w, F)\right)
\]
\[
= \mathbb{Pr}
\left(\bigvee_{\ell \in [m] \mid |V^\ell| \geq 2} \delta^\ell > rh(1 + \varepsilon) SC(w, F)\right)
\]
\[
\leq \sum_{\ell \in [m] \mid |V^\ell| \geq 2} \mathbb{Pr}(\delta^\ell > rh(1 + \varepsilon) SC(w, F))
\]
\[
< \frac{1}{m^\varepsilon} \cdot \frac{1}{m^\varepsilon} = \frac{1}{m^{2\varepsilon}} - 1.
\]
Since \(h \geq 1\), \(r \geq 2\) and since \(\delta^j \leq SC(w, F)\) for all \(\ell \in [m]\) with \(V^\ell = 1\) (by Lemma 5.4), we have
\[
\mathbb{Pr}
\left(\max_{\ell \in [m]} \delta^\ell > rh(1 + \varepsilon) SC(w, F)\right)
\]
\[
= \mathbb{Pr}
\left(\max_{\ell \in [m] \mid |V^\ell| \geq 2} \delta^\ell > rh(1 + \varepsilon) SC(w, F)\right)
\]
\[
\leq \frac{1}{m^{2\varepsilon}} - 1.
\]
\[
\sum_{0 \leq \delta \leq 2h(1+\varepsilon)SC(w,F)} \delta \Pr\left( \max_{\ell \in [m]} \delta^\ell = \delta \right) + \sum_{2 \leq r \leq \infty} \delta Pr\left( \max_{\ell \in [m]} \delta^\ell \leq 2h(1+\varepsilon)SC(w,F) \right) \\
\sum_{2 \leq r \leq \infty} \delta Pr\left( \max_{\ell \in [m]} \delta^\ell \leq (r+1)h(1+\varepsilon)SC(w,F) \right) \\
\leq 2h(1+\varepsilon)SC(w,F)Pr\left( \max_{\ell \in [m]} \delta^\ell \leq 2h(1+\varepsilon)SC(w,F) \right) \\
+ \sum_{2 \leq r \leq \infty} (r+1)h(1+\varepsilon)SC(w,F) \cdot Pr\left( \max_{\ell \in [m]} \delta^\ell > rh(1+\varepsilon)SC(w,F) \right) \\
< 2h(1+\varepsilon)SC(w,F) \cdot 1 \\
+ \sum_{2 \leq r \leq \infty} (r+1)h(1+\varepsilon)SC(w,F) \cdot \frac{1}{mr-1} \\
\left( \text{since } Pr\left( \max_{\ell \in [m]} \delta^\ell > rh(1+\varepsilon)SC(w,F) \right) < \frac{1}{mr-1} \right) \\
= 2h(1+\varepsilon)SC(w,F) + h(1+\varepsilon)SC(w,F) \cdot \frac{1}{m} \sum_{m \leq r \leq \infty} \frac{r+1}{mr-2} \\
= 2h(1+\varepsilon)SC(w,F) + h(1+\varepsilon)SC(w,F) \cdot O\left( \frac{1}{m} \right) \\
\left( \text{since } \sum_{2 \leq r \leq \infty} \frac{r+1}{mr} = O(1) \text{ for } m \geq 2 \right) \\
\leq 2h(1+2\varepsilon)SC(w,F),
\]

for suitable large \( m \). Hence,

\[
SC(w, P) = \mathcal{E}\left( \max_{\ell \in [m]} \delta^\ell \right) < 2h(1+2\varepsilon)SC(w,F)
\]

for any \( \varepsilon \), where \( 0 < \varepsilon < 1 \). This completes the proof of Theorem 5.1. □

If all user traffics are identical, that is, \( w_1 = w_2 = \ldots = w_n \), then \( h = \frac{mw_1}{w} = 1 \). Thus, Theorem 5.1 immediately implies:

**Corollary 0.1.** Consider the model of identical capacities. Let \( n = m, m \) suitable large. Then, for any traffic vector \( w \) with \( w_1 = w_2 = \ldots = w_n \) and Nash equilibrium \( P \), \( SC(w, P) < (2 + \varepsilon)SC(w, F) \), for any \( \varepsilon > 0 \).

Recall that there is a randomized, polynomial-time approximation scheme (RPTAS) to approximate the social cost of any Nash equilibrium (in particular, the fully mixed) within any arbitrary \( \varepsilon > 0 \) [7, Theorem 9]. Thus, since, by Theorem 5.1, the worst social cost is bounded by \( 2h(1 + \varepsilon) \) times the social cost of the fully mixed Nash equilibrium, this yields:
Theorem 5.5. Consider the model of identical capacities. Let \( n = m, m \) suitably large. Then, there exists a randomized, polynomial-time algorithm with approximation factor \( 2h(1 + \varepsilon) \), for any \( \varepsilon > 0 \), for WORST NASH EQUILIBRIUM SOCIAL COST.

We significantly improve Theorem 5.1 under a certain assumption on the traffics.

Theorem 5.6. Consider any traffic vector \( \mathbf{w} \) such that \( w_1 \geq w_2 + \cdots + w_n \). Then, for any Nash equilibrium \( \mathbf{P}, SC(\mathbf{w}, \mathbf{P}) \leq SC(\mathbf{w}, \mathbf{F}) \).

Proof. Since \( w_1 \geq w_2 + \cdots + w_n \), it follows that the link with maximum latency has user 1 assigned to it in any pure strategy profile. Thus, in particular, \( SC(\mathbf{w}, \mathbf{F}) = \lambda_1(\mathbf{w}, \mathbf{F}) \) and \( SC(\mathbf{w}, \mathbf{F}) = \lambda_1(\mathbf{w}, \mathbf{F}) \). By Lemma 4.1, \( \lambda_1(\mathbf{w}, \mathbf{P}) \leq \lambda_1(\mathbf{w}, \mathbf{F}) \). It follows that \( SC(\mathbf{w}, \mathbf{P}) \leq SC(\mathbf{w}, \mathbf{F}) \), as needed. \( \square \)

6. Best pure Nash equilibria and Nashification

We start by establishing \( NP \)-hardness for NASHIFY. Then we provide a polynomial-time algorithm to convert any pure strategy profile into a pure Nash equilibrium with non-increased social cost. Together with a PTAS for scheduling \( n \) jobs on \( m \) identical machines [11], this yields a PTAS for BEST PURE NASH EQUILIBRIUM.

Theorem 6.1. NASHIFY is \( NP \)-hard, even if \( m = 2 \).

Proof. By reduction from PARTITION. Consider any arbitrary instance of PARTITION consisting of a set \( A \) of \( k \) items \( a_1, a_2, \ldots, a_k \) with item sizes \( s(a_1), s(a_2), \ldots, s(a_k) \in \mathbb{N} \), for any integer \( k \). Construct from it an instance of NASHIFY as follows: Set \( n = 3k \) and \( m = 2 \). Set \( w_i = s(a_i) \) for \( 1 \leq i \leq k \), and \( w_i = 1/2k \) for \( k + 1 \leq i \leq 3k \). Take the pure strategy profile that assigns users 1, 2, \ldots, \( 2k \) to link 1 and users \( 2k + 1, \ldots, 3k \) to link 2.

We establish that this yields a reduction from PARTITION to NASHIFY. Assume first that the instance of PARTITION is positive; that is, there exists a subset \( A' \subseteq A \) such that \( \sum_{a \in A'} s(a) = \sum_{a \in A \setminus A'} s(a) \). Since either \( |A'| \leq k/2 \) or \( |A \setminus A'| \leq k/2 \), assume, without loss of generality, that \( |A'| \leq k/2 \). Note that each user assigned to link 1 is unsatisfied in the constructed pure strategy profile since its latency cost on link 1 is \( \sum_{a \in A} s(a) + k \cdot 1/2k = \sum_{a \in A} s(a) + 1/2 \), while its latency cost on link 2 is \( k \cdot 1/2k = 1/2 \), which is less. Thus, each step that transfers an unsatisfied user that corresponds to an element \( a \in A' \) from link 1 to link 2 is a selfish step, and the sequence of steps that transfer all users that correspond to elements of \( A' \) from link 1 to link 2 is a sequence of at most \( k/2 < k \) steps. As a result of this sequence of selfish steps, the latency of link 1 will be \( \sum_{a \in A} s(a) + 1/2 \), while the latency of link 2 will be \( \sum_{a \in A} s(a) + 1/2 \). Since \( \sum_{a \in A} s(a) = \sum_{a \in A \setminus A'} s(a) \), these two latencies are equal and the resulting pure strategy profile is therefore a Nash equilibrium, which implies that NASHIFY is positive.

Assume now that the instance of NASHIFY is positive; that is, there exists a sequence of at most \( k \) selfish steps that transforms pure strategy profile in the constructed instance of NASHIFY to a Nash equilibrium. Assume that in the resulting pure strategy profile
users corresponding to a subset $A' \subseteq A$ remain in link 1, users corresponding to the subset $A \setminus A' \subseteq A$ are transferred to 2, while the sums of traffics of users with traffic $1/2k$ that reside in links 1 and 2 are $\sum_{a \in A'} s(a) + x$ and $\sum_{a \in A \setminus A'} s(a) + 1 - x$, respectively. We consider two cases:

Assume first that $A' = A$. Then after at most $k$ selfish steps the latency on link 2 is at most 1 whereas the latency on link 1 is at least $\sum_{a \in A} s(a) \geq k$. So there exists an unsatisfied user $a \in A$, a contradiction to the fact that $\text{NASHIFY}$ is positive. So let $A' \neq A$. We show that this implies $\sum_{a \in A'} s(a) - \sum_{a \in A \setminus A'} s(a) = 0$. Assume $|\sum_{a \in A'} s(a) - \sum_{a \in A \setminus A'} s(a)| \neq 0$. Since the traffics of users in $A$ are integer, this implies $|\sum_{a \in A'} s(a) - \sum_{a \in A \setminus A'} s(a)| \geq 1$. The fact that $A' \neq A$ shows that at least one user with large traffic was transformed to link 2. So we can make at most $k - 1$ selfish steps with the small traffics. However, transforming $k - 1$ small traffics to the link with smaller latency leaves one user with small traffic unsatisfied, a contradiction to the fact that $\text{NASHIFY}$ is positive. So $|\sum_{a \in A'} s(a) - \sum_{a \in A \setminus A'} s(a)| = 0$, which implies that $\text{PARTITION}$ is positive. □

We remark that $\text{NASHIFY}$ is $\mathcal{NP}$-complete in the strong sense (cf. [9, Section 4.2]) if $m$ is part of the input. Thus, there is no pseudopolynomial-time algorithm for $\text{NASHIFY}$ (unless $\mathcal{P} = \mathcal{NP}$). In contrast, there is a natural pseudopolynomial-time algorithm $\text{A}_{k-\text{nashify}}$ for $k$-$\text{NASHIFY}$, which exhaustively searches all sequences of $k$ selfish steps; since a selfish step involves a (unsatisfied) user and a link for a total of $mn$ choices, the running time of $\text{A}_{k-\text{nashify}}$ is $\Theta((mn)^k)$. We continue to present an algorithm $\text{A}_{\text{nashify}}$ that solves $\text{NASHIFY}$ when $n$ selfish steps are allowed (Fig. 1).

The algorithm $\text{A}_{\text{nashify}}$ sorts the user traffics in non-increasing order so that $w_1 \geq \cdots \geq w_n$. Then for each user $i := 1$ to $n$, it removes user $i$ from the link it is currently assigned, it finds the link $\ell$ with the minimum latency, and it reassigns user $i$ to the link $\ell$.

The following lemma is crucial to prove the correctness of algorithm $\text{A}_{\text{nashify}}$.

**Lemma 6.2.** A greedy selfish step of an unsatisfied user $i$ with traffic $w_i$ makes no user $k$ with traffic $w_k \geq w_i$ unsatisfied.

**Proof.** Let $L = \langle l_1, \ldots, l_n \rangle$ be a pure strategy profile. Furthermore, let $p = l_i$ and let $q$ be the link with minimum latency. Denote $\lambda^j$ and $\hat{\lambda}^j$ the latency of link $j \in [m]$ before and
Theorem 6.4. There exists a

Let

Theorem 6.3.

to the move of user

after user \(i\) changed its strategy, respectively. Assume that user \(k\) becomes unsatisfied due to the move of user \(i\). Since only the latency on link \(p\) and \(q\) changed, we have to distinguish between two cases. Either \(l_k \neq q\) and user \(k\) wants to change its strategy to \(p\), or \(l_k = q\) and user \(k\) becomes unsatisfied due to the additional traffic \(w_i\) on link \(q\).

First, assume that \(l_k \neq q\), and that user \(k\) wants to change its strategy to \(p\). Since user \(i\) changed its strategy from \(p\) to \(q\) we know that \(\lambda^q < \lambda^p\) and therefore \(w_k + \lambda^q < w_k + \lambda^p\). So if user \(k\) wants to change its strategy to \(p\), then user \(k\) was already unsatisfied before user \(i\) changed its strategy, a contradiction.

For the case that the strategy of user \(k\) is \(q\) we define \(\lambda_q^q = \lambda^q - w_k\). We have \(\forall j \in [m]: \lambda_j^q + w_k \geq \lambda_j^q + w_i = \lambda_q^q + w_k + w_i\). Therefore \(k\) stays satisfied. \(\square\)

Theorem 6.3. Let \(L = (l_1, \ldots, l_n)\) be a pure strategy profile for \(n\) users with traffics \(w_1, \ldots, w_n\) on \(m\) links with social cost \(SC(w, L)\). Then algorithm \(\text{nashify}\) computes a Nash equilibrium from \(L\) with social cost \(\leq SC(w, L)\) in \(O(n \log n)\) time.

Proof. In order to complete the proof of Theorem 6.3, we have to show that algorithm \(\text{nashify}\) returns a pure strategy profile \(L'\) that is a Nash equilibrium and has social cost \(SC(w, L') \leq SC(w, L)\). It is easy to see that \(SC(w, L') \leq SC(w, L)\), since for user \(j\) we always choose the link with lowest latency as its strategy. After every iteration the user that changed its strategy is satisfied. Since we go through the list of users in descending order of their traffic and because of Lemma 6.2, all users that changed their strategy in earlier iterations stay satisfied. Therefore after we went through the complete list of users, all users are satisfied and thus \(L'\) is a Nash equilibrium.

The running time of algorithm \(\text{nashify}\) is \(O(n \log n)\) for sorting the \(n\) user traffics, \(O(m \log m)\) for constructing a heap with all latencies in the input pure strategy profile \(L\), and \(O(n \log m)\) for finding the minimum element of the heap in each of the \(n\) iterations of the algorithm. Thus, the total running time is \(O(n \log n + m \log m + n \log m)\). The interesting case is when \(m \leq n\) (since otherwise, a single user can be assigned to each link, achieving an optimal Nash equilibrium). Thus, in the interesting case, the total running time of \(\text{nashify}\) is \(O(n \log n)\). \(\square\)

Running the PTAS of Hochbaum and Shmoys [11] for scheduling \(n\) jobs on \(m\) identical machines yields a pure strategy profile \(L\) such that \(SC(w, L) \leq (1 + \varepsilon) \cdot OPT(w)\). On the other hand, applying the algorithm \(\text{nashify}\) on \(L\) yields a Nash equilibrium \(L'\) such that \(SC(w, L') \leq SC(w, L)\). Thus, \(SC(w, L') \leq (1 + \varepsilon) \cdot OPT(w)\). Since also \(OPT(w) \leq SC(w, L')\), it follows that:

Theorem 6.4. There exists a PTAS for BEST PURE NASH EQUILIBRIUM, for the model of identical capacities.

7. Worst pure Nash equilibria

In this section we consider worst pure Nash equilibria. We start by proving a tight upper bound on the social cost of any pure Nash equilibrium. Then, by reduction from BIN PACKING, we establish \(\mathcal{NP}\)-hardness for approximating a pure Nash equilibrium with
worst social cost within a factor better than $2 - 2/(m + 1)$. We close with a pseudopolynomial-
time algorithm to compute a worst pure Nash equilibrium if the number of links is fixed.

Denote with $m\text{-WCpNE}$ the decision problem corresponding to the problem to compute
the worst-case pure Nash equilibrium for $n$ users with traffics $w_1, \ldots, w_n$ on $m$ links. If $m$
is part of the input, then we call the problem $\text{WCpNE}$. We first show:

**Theorem 7.1.** Fix any traffic vector $w$ and pure Nash equilibrium $L$. Then, $\frac{\text{SC}(w, L)}{\text{OPT}(w)} \leq 2 - \frac{2}{m + 1}$. Furthermore, this upper bound is tight.

**Proof.** Schuurman and Vred eveld [29] showed the tightness of the upper bound for jump
optimal schedules proved by Finn and Horowitz [6]. Since every pure Nash equilibrium
is also jump optimal, the upper bound follows directly. Greedy selfish steps on identical
links can only increase the minimum load over all links. Thus, we can transform every
jump optimal schedule into a Nash equilibrium without altering the makespan, proving
tightness. $\square$

**Theorem 7.2.** It is $\mathcal{NP}$-hard to find a pure Nash equilibrium $L$ with $\frac{\text{WC}(w)}{\text{SC}(w, L)} < 2 - \frac{2}{(m + 1)} - \varepsilon$, for any $\varepsilon > 0$. It is $\mathcal{NP}$-hard in the strong sense if the number of links $m$ is part of the input.

**Proof.** We show that for a certain class of instances we have to solve $\text{BIN PACKING}$ in
order to find a Nash equilibrium with desired property. $\text{BIN PACKING}$ is $\mathcal{NP}$-complete in
the strong sense [9]. Consider an arbitrary instance of $\text{BIN PACKING}$ consisting of a set
of items $U = \{u_1, \ldots, u_{|U|}\}$ with sizes $s(u_j) \leq \delta$, $\sum_{u_j \in U} = m - 1$, and $K = m - 1$ bins
of capacity $B = 1$. From this instance we construct an instance for the stated problem as
follows: Set $\varepsilon = 2\delta$. There are $n - 2 = |U|$ users with traffic $w_i = s(u_i)$ and two users with
traffic $w_{n-1} = w_n = 1$. Note that the social cost of a Nash equilibrium is either 2 when the
users with traffic 1 are on the same link, or at most $(m + 1)/m + \frac{\varepsilon m}{2}$ otherwise.

If $\text{BIN PACKING}$ is negative, then there exists no Nash equilibrium with both users with
traffic 1 on the same link. Thus every Nash equilibrium has the desired property. If $\text{BIN PACKING}$ is positive, then there exists a Nash equilibrium with both users with traffic 1 on
the same link. The social cost of this Nash equilibrium is $\text{WC}(w) = 2$. For any other Nash
equilibrium $L$ where the users with traffic 1 use different links, $\text{SC}(w, L) \leq (m + 1)/m + \delta$.

This yields

$$\frac{\text{WC}(w)}{\text{SC}(w, L)} \geq \frac{2}{m + 1 + \delta} = \frac{2}{m + 1 + \frac{\varepsilon m}{2}} = \frac{2m}{m + 1 + \frac{\varepsilon m}{2}}$$

$$= 2 - \frac{2}{m + 1 + \frac{\varepsilon m}{2}} - \frac{\varepsilon m}{m + 1 + \frac{\varepsilon m}{2}}$$

$$> 2 - \frac{2}{m + 1} - \varepsilon.$$ 

So, to find a Nash equilibrium with desired property, we have to find a distribution of the
small traffics $w_1, \ldots, w_{n-2}$ to $m - 1$ links which solves $\text{BIN PACKING}$. 

Since BIN PACKING is $\mathcal{NP}$-hard in the strong sense, if the number of bins is part of the input, it follows that computing a pure Nash equilibrium $L$ with $\text{WC}(w)/\text{SC}(w, L) < 2 - 2/(m + 1) - \varepsilon$ is also $\mathcal{NP}$-hard in the strong sense, if $m$ is part of the input. □

Since $\text{WCpNE}$ is $\mathcal{NP}$-hard in the strong sense [7], there exists no pseudopolynomial algorithm to solve $\text{WCpNE}$. However, we can give such an algorithm for $m$-$\text{WCpNE}$.

**Theorem 7.3.** There exists a pseudopolynomial-time algorithm for $m$-$\text{WCpNE}$.

**Proof.** We start with the state set $S_0$ in which all links are empty. After inserting the first $i$ traffics, the state set $S_i$ consists of all $(2m)$-tuples $(\lambda_1, \tilde{w}_1, \ldots, \lambda_m, \tilde{w}_m)$ describing a possible placement of the largest $i$ traffics with $\lambda_j$ being the latency on link $j$ and $\tilde{w}_j$ the smallest traffic placed on link $j$. We need at most $m \cdot |S_i|$ steps to create $S_{i+1}$ from $S_i$, and $|S_i| \leq (W_i)^m \cdot (w_1)^m$, where $W_i = \sum_{j=1}^{i} w_j$. Therefore the overall computation time is bounded by $O(n \cdot m \cdot W^m \cdot (w_1)^m)$. The best-case Nash equilibrium and the worst-case Nash equilibrium can be found by exhaustive search over the state set $S_n$ using $O(n \cdot m \cdot W^m \cdot (w_1)^m)$ time. □

**Remark.** Theorem 7.3 also holds for the case of arbitrary link capacities.

8. Conclusions and discussion

In this work, we have studied the combinatorial structure and the computational complexity of the extreme (either worst or best) Nash equilibria for the selfish routing game introduced in the pioneering work of Koutsoupias and Papadimitriou [16].

Our study of the combinatorial structure has revealed an interesting, highly non-trivial, combinatorial conjecture about the worst such Nash equilibrium, namely the Fully Mixed Nash Equilibrium Conjecture, abbreviated as FMNE Conjecture; the conjecture states that the fully mixed Nash equilibrium [19] is the worst Nash equilibrium in the setting we consider. We have established that the FMNE Conjecture is valid when restricted to pure Nash equilibria. Furthermore, we have come close to establishing the FMNE Conjecture in its full generality by proving that the social cost of any (pure or mixed) Nash equilibrium is within a factor of $2h(1 + \varepsilon)$, for any $\varepsilon > 0$, of that of the fully mixed Nash equilibrium, where $h$ is the factor by which the largest user traffic deviates from the average user traffic, and under the assumptions that all link capacities are identical, the number of users is equal to the number of links and the number of links is suitably large. The proof of this result has relied very heavily on applying and extending techniques from the theory of stochastic orders and majorization [18,30]; such techniques are imported for the first time into the context of selfish routing, and their application and extension are both of independent interest. We hope that the application and extension of techniques from the theory of stochastic orders and majorization will be valuable to further studies of the selfish routing game considered in this paper and for the analysis and evaluation of mixed Nash equilibria for other games as well.

Our study of the computational complexity of extreme Nash equilibria has resulted in both positive and negative results. On the positive side, we have devised, for the case of
identical link capacities, equal numbers of users and links and a suitably large number of links, a randomized, polynomial-time algorithm to approximate the worst social cost within a factor arbitrarily close to $2h(1 + \varepsilon)$, for any $\varepsilon > 0$. The approximation factor $2h(1 + \varepsilon)$ of this randomized algorithm will immediately improve upon reducing $2h$ further down in our combinatorial result described above, relating the social cost of any Nash equilibrium to that of the fully mixed. We have also introduced the technique of Nashification as a tool for converging to a Nash equilibrium starting with any assignment of users to links in a way that does not increase the social cost; coupling this technique with a polynomial-time approximation scheme for the optimal assignment of users to links [11] has yielded a polynomial-time approximation scheme for the social cost of the best Nash equilibrium. In sharp contrast, we have established a tight limit on the approximation factor of any polynomial-time algorithm that approximates the social cost of the worst Nash equilibrium (assuming $P \neq NP$).

Our approximability and inapproximability results for the best and worst Nash equilibria, respectively, establish an essential difference between the approximation properties of the two types of extreme Nash equilibria.

The most obvious problem left open by our work is to establish the FMNE Conjecture. Some progress on this problem has been already reported by Lücking et al. [17], where the conjecture is proved in various special cases of the model of selfish routing introduced by Koutsoupias and Papadimitriou [16] and considered in this work; furthermore, Lücking et al. disprove the FMNE Conjecture in a different model for selfish routing that borrows from the model of unrelated machines [12] studied in the scheduling literature.

The technique of Nashification, as an algorithmic tool for the computation of Nash equilibria, also deserves further study. Some steps in this direction have been taken already by Feldmann et al. [5].

Establishment of the Fully Mixed Nash Equilibrium Conjecture will reveal an interesting complexity-theoretic contrast between the worst pure and mixed Nash equilibria. On the one hand, computing the (supports of the) worst pure Nash equilibrium is an $NP$-hard problem [7, Theorem 4]; however, computing the social cost of a worst pure Nash equilibrium is trivially in $P$ (since it amounts to computing the maximum). On the other hand, if the fully mixed Nash equilibrium conjecture is true, computing the supports of a worst mixed Nash equilibrium is a trivial problem and, moreover, the polynomial characterization of the fully mixed Nash equilibrium shown in [19, Theorem 14] implies that a worst mixed Nash equilibrium can be computed in polynomial time; however, computing the social of a worst mixed Nash equilibrium remains $\#P$-complete. This result follows from an inspection of the proof of [7, Theorem 8], which establishes that computing the social cost of a Nash equilibrium is a $\#P$-complete problem. We consider this different behavior of pure and mixed Nash equilibria to be an interesting complexity-theoretic consequence of the Fully Mixed Nash Equilibrium Conjecture.

Acknowledgements

We would like to thank Rainer Feldmann and Manuel Rode for many fruitful discussions. We are also very grateful to Petra Berenbrink and Tasos Christophides for many helpful discussions on stochastic orders. Moreover, we are much obliged to Andreas Baltz for discussions on Theorem 5.1.
References


