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2005
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Abstract. We consider size approximation problem for single-hop radio networks. Unlike other algorithms available in the literature we consider not only estimation with an accuracy within a constant factor, but also take care of the bias of the estimation. Unexpectedly, it turns out that estimations based on medium of values obtained in independent subgroups provide much better results than analogous estimations based on mean values. Our approach yields an algorithm with a low runtime and energy cost. It can be also tuned to be robust against random communication failures and against an adversary with limited energy resources.

1 Introduction

Single-Hop Radio Network Model. A radio network consists of processing units called stations or processors, which can either send or receive messages through a shared communication channel. The stations have synchronized clocks, execution time is divided into slots. During a time slot a station may either send or receive, but not both. A station may also switch off its transmitter/receiver. A message sent by a single station can be received by any other station of the network (single-hop network); if two or more stations send simultaneously, then a collision occurs and no message comes through.

The most important complexity measures for algorithms on radio networks are runtime and energy cost, which is the maximum over all stations of the number of slots during which its transmitter/receiver is switched on. Energy cost is closely related to maximum of real energy usage of devices. It is quite important, since the devices are battery operated in most cases.

Size Approximation Problem. Before running any other algorithm, it is often necessary to find the size of a radio network, which is the number of active stations. Computing the exact number of stations might be costly (the runtime is at least \( \Theta(N) \) for a network with \( N \) stations). However, very often we need only an approximation: we look for a number \( n \) such that \( \frac{n}{\alpha} \leq N \leq n \cdot \alpha \), where \( \alpha \) is a chosen approximation factor.

Size approximation problem was concerned in many papers, see for instance [1, 3, 8, 7]. The most efficient size approximation algorithm was developed in [2]. Its time complexity is \( O(\log \log N) \) and energy cost is \( O((\log N)^2 + \epsilon) \). The main problem is that it works only if communication is reliable. Namely, if a single message is sent, then

* Partially supported by the European Union within the 6th Framework Programme under contract 001907 (DELIS).
no error occurs: it will be received by every station which has its receiver switched on. Also, bias of the approximation has not been estimated – it was only guaranteed that the estimated size is within a constant factor of the real value.

Our goal is to develop algorithmic tools such that the algorithm becomes both robust against communication failures (or adversarial attacks) and provides an estimation with a small bias. Due to space limitations, we shall discuss only the issues of a low bias.

2 Size approximation algorithm

Assume that there are \( N \) stations in the network. The algorithm consists in stages executed sequentially. In a stage \( i \) we assume that \( N \in (2^i, 2^{i+1}) \) - if this assumption is false, then with high probability no estimation is found and the next round starts. Otherwise, with a high probability we get a size estimation.

Now let us consider a stage \( i \). We arrange the stations in \( k = 2^i \) groups, where each group consists in \( k \) subgroups. For each subgroup a station joins this group with probability \( 2^{-2i} \), independently from the decisions regarding other subgroups. So a station can belong to several subgroups as well as to none. (We can adopt a different assignment to groups, as in [2], this would guarantee a low energy cost, but would result in a more complex analysis.) So a random variable denoting the number of stations within a given subgroup has the Bernoulli distribution with parameters \( N \) and \( 2^{-2i} \).

There are 8 time slots devoted to each subgroup. For each slot, every station assigned to the subgroup concerned transmits with a probability depending on the subgroup index. For the \( j \)th subgroup in a group this probability is \( \frac{1}{3} \cdot 2^{-(2^i - 2i - j)} \).

If a single station transmits in a time slot, we call such an event \( SINGLE \). It is widely known that probability of \( SINGLE \) is maximized if the number of stations assigned to a subgroup is \( \frac{1}{p} \), where \( p \) is the broadcast probability [2]. If this is the case, the expected number of \( SINGLEs \) in the 8 time slots for such a subgroup equals \( \frac{8}{3} \approx 3 \). Let \( 3SINGLES \) denote the event that there are 3 \( SINGLEs \) for a given subgroup. Therefore we seek subgroups with \( 3SINGLES \), and each of them corresponds to a broadcast probability which is roughly the inverse of the number of stations in the subgroup.

In the next phase, the goal is that all stations in a group learn about all \( 3SINGLES \) occurred in the group. We can achieve this in a short time and small energy cost: every subgroup elects its leader, which is the station which has sent the first \( SINGLE \). Then for a parameter \( \beta \) (practically \( \beta \) can be a small constant) a leader of subgroup \( j \) with \( 3SINGLES \) transmits at time slot \( (j \text{ mod } \beta) + 1 \) informing about \( j \) and all subgroups with \( 3SINGLES \) learned so far in the group. All stations of the group listen during all \( \beta \) steps.

Now, for each group where \( 3SINGLES \) have occurred all stations assigned to it have the same information on the subgroups with \( 3SINGLES \). Based on the broadcast probability in the subgroups with \( 3SINGLES \) the stations construct an estimate for stations quantity within those subgroups; for broadcast probability \( p \) this estimate is \( \frac{1}{p} \).

Each station forms a list of these estimates - each entry has a field indicating a group number, one entry per subgroup with \( 3SINGLES \). Now, the stations collect information from other groups. For this purpose a gossiping procedure is started. A simplest way
is that at step \( i \) a station informs about all 3SINGLES up to group \( i - 1 \). The station transmitting is a leader of a group \( j \) with 3SINGLES, where \( j < i \) is maximal with 3SINGLES. If group \( i \) has 3SINGLES, then its leader confirms receiving the message (and existence of 3SINGLES in group \( i \)).

After step \( k \) the last group with 3SINGLES transmits the information on all subgroups with 3SINGLES, all stations listen. Now the final estimate of network size is based

- either on the mean of the estimates from subgroups
- or the median of the list of estimates.

These values are multiplied with \( k^2 \) to form the final estimate (this is necessary, since each of the estimates concerns the size of a subgroup).

It can be easily seen that the algorithm runtime is \( O(\log^2 N \cdot \log \log N) \) and its energy cost is \( O(\log \log N) \). If we would like to make it immune against communication failures and collisions caused by an adversary, then some modifications in the last phases are necessary (see [4]).

### 3 Analysis of Algorithm Behavior

Let \( X \) be the number of stations in a subgroup. Then

\[
X \sim B(N, 2^{-2i}), \quad E[X] = N \cdot 2^{-2i}, \quad Var[X] = N \cdot 2^{-2i} \cdot (1 - 2^{-2i}) \approx E[X].
\]

Therefore, from the Chebyshev inequality we have, for every \( c \in (0, 1) \):

\[
Pr[|X - EX| > (1 - c) \cdot EX] < \frac{Var[X]}{(1 - c) \cdot E[X]^2} \approx \frac{1}{(1 - c) \cdot E[X]^2}.
\]  

On the other hand, for every \( c \) and \( \epsilon \) for almost all \( N \)

\[
\frac{1}{(1 - c) \cdot E[X]} < \epsilon.
\]

Consider now a single subgroup. Let \( n \) be such that \( 2^n < N \leq 2^{n+1} \). So \( \frac{2^n}{N^2} \leq E[X] = N \cdot 2^{-2i} \leq \frac{2^{n+1}}{N^2} \). If \( X = x \), then probability of a SINGLE in one slots is equal to \( x \cdot p \cdot (1 - p)^{x-1} \), where \( p \) is the broadcast probability in this subgroup. Therefore, for this subgroup

\[
Pr[\text{3SINGLES}] = \binom{N}{x} \cdot (x \cdot p \cdot (1 - p)^{x-1})^3 \cdot (1 - x \cdot p \cdot (1 - p)^{x-1})^5.
\]

Based on (1) we assume that \( c \cdot E[X] \leq x \leq (2 - c) \cdot E[X] \) with the probability \( 1 - \epsilon \). Then the probability of 3SINGLES for the subgroup equals

\[
\sum_{N=2^n}^{2^{n+1}} \sum_{x=cE[X]}^{(2-c)E[X]} \binom{N}{x} \cdot (x \cdot p \cdot (1 - p)^{x-1})^3 \cdot (1 - x \cdot p \cdot (1 - p)^{x-1})^5 \cdot \left( \frac{1}{N^2} \right)^x \cdot \left( 1 - \frac{1}{N^2} \right)^{N-x}.
\]  

where

\[
\binom{N}{x} \left( \frac{1}{N^2} \right)^x \left( 1 - \frac{1}{N^2} \right)^{N-x}.
\]
corresponds to the assumption that every \( N \) between \( 2^n \) and \( 2^{n+1} \) is equally probable.

Let us now assess that probability given by (2). We estimate probabilities of 3SINGLES in five subgroups: the one with broadcast probability closest to \( \frac{k^2}{2^n} \) and in four neighboring subgroups. We assess those probabilities by calculating the worst and the best case in every subgroup depending on the relation between \( N \) and \( n \). The calculations have asymptotic character. Let us first consider the optimal subgroup, that is, the centre of the interval).

In the first case, the worst \( x \) we can have equals \( c \cdot \mathbb{E}[X] = c \cdot \frac{2^n}{k^2} \). Therefore,

\[
\Pr[3\text{SINGLES}] > \binom{N}{3} \left( c \cdot \frac{2^n}{k^2} \right)^3 \left( 1 - \frac{k^2}{3 \cdot 2^n - 1} \right)^3 \left( 1 - c \cdot \frac{2^n}{3 \cdot 2^n - 1} \right)^5 \left( 1 - \frac{k^2}{3 \cdot 2^n - 1} \right)^5
\]

\[
\approx 56 \left( \frac{2^n}{3} \left( \frac{1}{c} \right)^{2^n} \right)^3 \left( 1 - \frac{2^n}{3} \left( \frac{1}{c} \right)^{2^n} \right)^5.
\]

In the second bad case, the worst \( x \) we can have is: \( (2 - c) \cdot \mathbb{E}[X] = (2 - c) \cdot \frac{2^{n+1}}{k^2} \). Therefore,

\[
\Pr[3\text{SINGLES}] > \binom{N}{3} \left( (2 - c) \cdot \frac{2^{n+1}}{k^2} \right)^3 \left( 1 - \frac{k^2}{3 \cdot 2^n - 1} \right)^3 \left( 1 - (2-c) \cdot \frac{2^{n+1}}{k^2} \right)^5 \left( 1 - \frac{k^2}{3 \cdot 2^n - 1} \right)^5
\]

\[
\approx 56 \left( \frac{4(2-c)}{3} \left( \frac{1}{c} \right)^{4(2-c)} \right)^3 \left( 1 - \frac{4(2-c)}{3} \left( \frac{1}{c} \right)^{4(2-c)} \right)^5.
\]

On the other hand, the best situation is when \( N = 3 \cdot 2^{n-1} \) and \( x = \mathbb{E}[X] = \frac{3 \cdot 2^{n-1}}{k^2} \). In this case

\[
\Pr[3\text{SINGLES}] < \binom{N}{3} \left( \frac{3 \cdot 2^{n-1}}{k^2} \right)^3 \left( 1 - \frac{k^2}{3 \cdot 2^n - 1} \right)^3 \left( 1 - \frac{k^2}{3 \cdot 2^n - 1} \right)^5
\]

\[
\approx 56 \left( \frac{1}{c} \right)^3 \left( 1 - \frac{1}{c} \right)^5.
\]

Similarly, we calculate the assessments for the probability of the 3SINGLES in the neighboring groups; the worst case in the groups with the broadcast probability less than \( \frac{k^2}{3 \cdot 2^n - 1} \) is when \( N = 2^{n+1} \) and \( x = (2 - c)\mathbb{E}[X] \) whereas in the groups where broadcast probability is greater than \( \frac{k^2}{3 \cdot 2^n - 1} \), the worst situation is when \( N = 2^n \) and
\( x = c \cdot E[X] \). Taking this into consideration one can easily derive assessments for the asymptotic probabilities of 3SINGLES in these groups.

Asymptotically, when \( c \to 1 \) the probability in the neighboring groups, in the worst cases will increase while in the best cases will decrease; of course, in the optimal group, the probability in the worst cases will increase and in the best case will remain the same. Therefore, taking \( c = 1 \) we obtain infimum and supremum for the probabilities in specific groups. These results are presented below in Table 3.

<table>
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<tr>
<th>subgroup broadcast probability</th>
<th>( N )</th>
<th>( c )</th>
<th>( \frac{3}{9} )</th>
<th>( \frac{9}{99} )</th>
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<td>( 2^{n+1} )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>( \frac{k^3}{3 \cdot 2^{2i+j}} )</td>
<td>( 2^n )</td>
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<td>.0028</td>
<td>.0009</td>
<td>.0008</td>
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<tr>
<td>( \frac{k^3}{3 \cdot 2^{2i+j}} )</td>
<td>( 2^{n+1} )</td>
<td>0</td>
<td>0.002</td>
<td>0.0007</td>
<td>0.0008</td>
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<tr>
<td>( \frac{k^3}{3 \cdot 2^{2i+j}} )</td>
<td>( 2^{n} )</td>
<td>.2292</td>
<td>.1693</td>
<td>.1319</td>
<td>.1279</td>
</tr>
<tr>
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<td>( 2^{n+1} )</td>
<td>.0500</td>
<td>.0912</td>
<td>.1239</td>
<td>.1279</td>
</tr>
<tr>
<td>( \frac{k^3}{3 \cdot 2^{2i+j}} )</td>
<td>( 2^{n} )</td>
<td>.2814</td>
<td>.2807</td>
<td>.2792</td>
<td>.2789</td>
</tr>
<tr>
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<td>( 2^{n+1} )</td>
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<td>.2814</td>
<td>.2787</td>
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<td>.2814</td>
<td>.2814</td>
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<tr>
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<td>.1931</td>
<td>.1949</td>
</tr>
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<td>.2791</td>
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<td>.2764</td>
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<td>.0604</td>
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<td>.0735</td>
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<td>.0735</td>
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</table>

### 3.1 Bias of the estimate

Let \( ‘N \leftrightarrow j’ \) denotes the case when the \( j \)th subgroup is the optimal one. Therefore, the mean bias of the estimate is given by the following formula:

\[
\sum_{j=1}^{2^i} \Pr(N \leftrightarrow j) \cdot E[\text{estimate’s bias assuming } N \leftrightarrow j] 
\]

For \( j \leq 2^i \), the \( j \)th subgroup corresponds to the interval \([2^j-j^{-1}, 2^{-j}]\). So the probability that \( N \leftrightarrow j \) is given by:

\[
\frac{2^j-j^{-1}}{2^{2^i+1}-2^{2^i}} = \frac{2^{j-1}}{2^{2^i-1}}.
\]

The mean bias of estimate assuming \( N \leftrightarrow j \) can be calculated in the following way:

\[
\frac{1}{2^{2^i+j} + 1} \sum_{N=2^{2^i+j}}^{2^{2^i+j+1}} \frac{\Pr(3\text{SINGLES in subgroup } l[N \leftrightarrow j])}{N} - 3 \cdot 2^{2^i-2+j}
\]

5
In order to assess (4) and (5) we use the following facts:

- The probability of 3SINGLES in the subgroup with broadcast probability $2^3$ times lesser than in the optimal group is approximately 0.

  Proof. Based on the probabilities assessments in the previous paragraph and taking $c=9$ we obtain that probability of 3SINGLES in the subgroup that has broadcast probability 8 times lesser than in the optimal group is less than $3 \cdot 10^{-3}$.

- The probability of 3SINGLES in the group with broadcast probability $2^5$ times lesser than in the optimal subgroup is approximately 0.

  Proof. Based on the probabilities assessments in the previous paragraph and taking $c=9$ we obtain that probability of 3SINGLES in the subgroup that has broadcast probability 8 times lesser than in the optimal group is less than $3 \cdot 8 \cdot 10^{-3}$.

- The probability that $N$ corresponds to one of the last four subgroups in a group equals

\[
\frac{2^{2^i-1}}{2^{2^i-1}} + \frac{2^{2^i-2}}{2^{2^i-1}} + \frac{2^{2^i-1}}{2^{2^i-1}} + \frac{2^{2^i-1}}{2^{2^i-1}} \approx 0.9375.
\]

Therefore, when assessing (3), we may consider only the last four subgroups as candidates for being the optimal ones.

- Assessing asymptotic bias we may take $N = 3 \cdot 2^i-2+j$ instead of $\sum_{i=j+1}^{N} 2^i$ without significant lost of accuracy. Therefore,

\[\Pr[3\text{SINGLES in subgroup } j-2] \approx 56 \left(4e^{-4}\right)^3 \left(1-4e^{-4}\right)^5 \approx 0.0151,\]
\[\Pr[3\text{SINGLES in subgroup } j-1] \approx 56 \left(2e^{-2}\right)^3 \left(1-2e^{-2}\right)^5 \approx 0.2292,\]
\[\Pr[3\text{SINGLES in subgroup } j] \approx 56e^{-3} \left(1-e^{-1}\right)^5 \approx 0.2814,\]
\[\Pr[3\text{SINGLES in subgroup } j+1] \approx 56 \left(2^{-1}e^{-\frac{1}{2}}\right)^3 \left(1-2^{-1}e^{-\frac{1}{2}}\right)^5 \approx 0.2564,\]
\[\Pr[3\text{SINGLES in subgroup } j+2] \approx 56 \left(4^{-1}e^{-\frac{1}{4}}\right)^3 \left(1-4^{-1}e^{-\frac{1}{4}}\right)^5 \approx 0.14,\]
\[\Pr[3\text{SINGLES in subgroup } j+3] \approx 56 \left(8^{-1}e^{-\frac{1}{8}}\right)^3 \left(1-8^{-1}e^{-\frac{1}{8}}\right)^5 \approx 0.0419.\]

We take into account above-mentioned facts, after short calculation we obtain that mean percentage error of quantity estimate is equal 16.09%. However, the estimate overestimate the real quantity in a case that the optimal subgroup is different from the last one. When $N \rightarrow 2^i$, the estimate underestimates $N$. That is why we also calculate the absolute percentage error of proposed estimate which is equal 40%.

Bias for Estimation Based on Median The probability of 3SINGLES in given subgroup is no symmetrical according to optimal subgroup. This is the reason for the bias of the estimate. In order to decrease such a bias, mean over vector $v \cdot k^2$ can be replaced by its median. As before, mean bias of the estimate based on median is given by the equation (3). However, now the estimations, and so the expected biases are computed differently. Since almost always 3SINGLES occur only in 6 subgroups (the optimal one, two below and three above the optimal), we consider only these subgroups. Therefore, there are 21 possible medians (6 from the subgroups and 15 for the arithmetic mean when the list of
estimations has even length). Let vector \( m \) denote these possible medians. Therefore, the equation (5) for estimate based on median is as follows:

\[
\frac{1}{2^{2^{i}+j} + 1} \sum_{N=2^{2^{i}+j}}^{2^{2^{i}+j+1}} N - \sum_{l=1}^{2^{i}} \Pr[\text{median} = m(l)|N \leftarrow j] \cdot m(l) .
\]

(6)

Similarly to the previous estimate, we may derive the following final assessment for the mean bias for the estimate based on median:

\[
\sum_{j=2^{i-4}}^{2^{i}} \frac{2^{j-1} \cdot 3 \cdot 2^{2^{i}+j} - \sum_{l=1}^{2^{i}} \Pr[\text{median} = m(l)|N \leftarrow j] \cdot m(l)}{3 \cdot 2^{2^{i}-2^{j}}} .
\]

(7)

After a short calculation we obtain that mean percentage error of quantity estimate is round 1.50% whereas the absolute percentage error of the estimate is round 18%. These results are significantly better than the ones corresponding to the estimate based on mean.

Finally, we compare those assess for both estimates in the best and the worst cases for particular values of \( N \). The results are listed in Table 3.1.

Finally, the probability there will be no 3SINGLES in the subgroups concerned can be assessed by the probability there will be no 3SINGLES within the optimal group as well as within group with 2 times lesser broadcast probability:

\[
\Pr[\text{no 3SINGLES}] < (1 - .2814)^{2^i} (1 - .2292)^{2^i} \rightarrow 0 \text{ as } i \rightarrow \infty .
\]

3.2 Simulations

In the simulations we assume that \( N \in [2^8, 2^{16}] \). In this case there are 8 groups, each consisting in 8 subgroups. We perform 100000 simulation. In each one, \( N \) is uniformly chosen from the set \([2^8, 2^{16}]\), stations chose subgroups and afterwards broadcasting phase take place. We search for subgroups with 3SINGLES and based on their broadcast probabilities we construct an estimate for \( N \). For each simulation, the estimates based on mean, median and mode are constructed and percentage error as well as absolute percentage error for all of the estimates are calculated. The results as follows:

There were no 3SINGLES in none of the subgroups in 0.7% of cases. Empirical experiments confirm theoretic assessments for bias. We can state that estimate based on median performs significantly better than the one based on mean.

References

<table>
<thead>
<tr>
<th>case</th>
<th>Pr[N \leftarrow\text{given subgroup}]</th>
<th>N</th>
<th>N_{\text{mean}}</th>
<th>N_{\text{median}}</th>
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<td></td>
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<td></td>
<td>2^{2^i+2}</td>
<td>-59.7</td>
<td>-31.47</td>
</tr>
<tr>
<td>N \leftarrow 2^i</td>
<td>\frac{2^{2^i-1}}{2^{2^i-1}}</td>
<td>2^{2^i+1}</td>
<td>-16.68</td>
<td>-7.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2^{2^i+1}</td>
<td>-30.97</td>
<td>-24.66</td>
</tr>
</tbody>
</table>

Table 1. Estimate's error in different subgroups.

<table>
<thead>
<tr>
<th>mean error(%)</th>
<th>median</th>
<th>mean absolute error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-11.48</td>
<td>-2.26</td>
<td>35.63</td>
</tr>
<tr>
<td>31.64</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Comparison of estimates based on mean an median.