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Statistical properties of directed ad-hoc networks arising from random distributed transmitting powers of agents

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I. INTRODUCTION

The last two decades have witnessed a vertiginous increase of wireless devices in correspondence to an increased overall human mobility. Almost every family in western cultural world possesses at least one mobile phone (MP). As today, multiple MPs inside the same family is a common feature rather than the exception. In parallel to MP, the development of wireless local area networks (WLAN) at relatively low cost together with competitive digital subscriber line (DSL) connection costs have favored the multiply of local wireless networks. The high costs of cabling needed by conventional wired LANs is no more justified by the case and speed of installation of WLANs. Very recently the implementation of the voice over IP protocol (VoIP), by which it is possible to use a LAN connection to transfer voice in real time so to use our own personal computer as telephone, has given and will give a further push to WLAN deployment. A notebook may be used as telephone at very low fares if a WLAN is around, so to formally substitute MPs. It is even already possible to find on the market wireless telephones that connect to WLANs without the need of a personal computer.

It is than evident that the future goal of local institutions and companies will be to reach the largest coverage of wireless signal, shall it be used by MP or WLAN, with the lowest economic cost. Nowadays the politics of communication companies is to ensure the most coverage by using MP base station antennas placed in strategic places on the territory. Coverage might be also provided by satellites in those places where it is suddenly needed as in military operation zones into enemy borders or in hardly accessible environments as deserts or high mountains. The use of base station antennas to provide signal in the territory is directly bound to human health. The consensus of the scientific community, both in the US and internationally, is that the power from these mobile phone base station antennas is far too low to produce health hazards as long as people are kept away from direct access to the antennas. There are some circumstances under which an improperly designed (or inadequately secured) mobile phone base station site could fail to meet safety guidelines. Safety guidelines for uncontrolled public exposure could be exceeded if antennas were mounted in such a way that the public could gain access to areas within 8 meters (horizontal) of the radiating surface(s) of the antennas themselves. This could arise for antennas mounted on or near the roofs of buildings. For example, Petersen et al. [3] found that 1 meter from a rooftop antenna radiating 1600 W effective radiating power (ERP), the power density was as high as 2 mW/cm^2 (compared to the ANSI/IEEE [4] public exposure standard of 0.57-1.2 mW/cm^2). For antennas mounted on towers, it is somewhat difficult to imagine a situation that would not meet the safety guidelines. However, there are reports (principally from outside North America and Europe) of mobile phone base station antennas facing directly at nearby buildings. Whether these antennas would meet the safety guidelines would depend on the ERP, the exact geometry and the degree of shielding provided by the building. In addition to human health problems, base station antennas need continuous maintenance service and the black out of just one of them generally uncovers large portions of territory.

An alternative to base station antennas that might partially solve the previously cited problems, might be achieved by means of a special kind of geometrical networks known as ad-hoc networks. Connectivity of clients in an ad-hoc network is reached by means of signal transmitters scattered in the territory. Differently from base station antennas, transmitters of ad-hoc networks have a much lower transmitting power (e.g. up to 2W ERP), while the signal coverage of territory is ensured by their large number. Each transmitter emits with a given radi-
transmitters in an ad-hoc complex topological network arising by connecting signal and in the following, we shall assume that the signal is carried their devices with them, so that the center of the effective fraction of public users. Clearly, those users would clarify that transmitters might be delivered to a certain position at random in a given box. An ad-hoc network will be built associating a node with a single user and drawing links whenever transmitter centered circles with given effective radii overlap. From now on, we shall use differently the denomination of node, user, and transmitter. The time evolution of the network will be mimicked by different ensemble realizations of the system so that the time averaged properties of the system will be scrutinized by ensemble averages. Transmitters will be placed at random in a two-dimensional unit box with both horizontal and vertical coordinates extracted with uniform probability. Periodic boundary conditions will be used, since it was shown that, with the same finite number of nodes, they deliver more accurate results with respect to sharp boundaries [9]. Although the real situation is of course three-dimensional, we perform our analysis in two dimensions, restricting ourselves to the case of the average transmitting radius larger than usual building heights. A transmitting radius will be assigned to each of the transmitters. The undirected network arising when transmitting radii are all equal to a given value and two nodes are considered connected if their associated circle overlap, is often in the literature referred to as unit disk graph [2]. We shall go further and also analyze the properties of the directed network constructed by extracting the effective transmission radius of each node from a given probability distribution with finite support. Links in this directed network will be drawn from a transmitter towards all other transmitting devices lying inside its effective covered area. This kind of network would be very close to what we would get in reality.

A. Total transmitting power

We shall show here with the help of simple considerations that the total electromagnetic power necessary to cover with a signal a given area $A$ embedded in a three-dimensional space, is independent from the effective transmitting radius of transmitters. We shall see that this property stems from the fact that both the number of transmitters necessary to cover a certain area, and the radiated power per surface unit in three dimensions, scale with the inverse square of the effective transmitting radius. Let us indicate with $P_i$ the effective radiating power of a transmitter necessary to maintain power $W_i$ at a distance $r_i$ from the source. At distance $r_i$ from the transmitter the power $W_i$ measured by a receiving device is simply proportional to $P_i/r_i^2$ [10]. Suppose a receiver is able to detect signals only above a certain threshold $T$, which then defines the effective transmission radius

II. GENERAL CONSIDERATIONS

In the following we shall analyze the properties of the complex topological network arising by connecting signal transmitters in an ad-hoc fashion as introduced above. Our final aim is to study the connectivity properties of such a network. In the introductory part of this work we clarified that transmitters might be delivered to a certain fraction of public users. Clearly, those users would carry their devices with them, so that the center of the effective transmitting sphere would wander in space. Here and in the following, we shall assume that the signal is transmitted isotropically, i.e. the lines at constant radiating power are simply spheres. The spatial position of transmitters would then be bound to the spatial position of users. Although it is in principle possible to distinguish from different typologies of users (travel salesmen or housewives for instance, the former with high mobility, the latter more static), we shall consider their position at random in a given box. An ad-hoc network will be built associating a node with a single user and drawing links whenever transmitter centered circles with given effective radii overlap. From now on, we shall use differently the denomination of node, user, and transmitter. The time evolution of the network will be mimicked by different ensemble realizations of the system so that the time averaged properties of the system will be scrutinized by ensemble averages. Transmitters will be placed at random in a two-dimensional unit box with both horizontal and vertical coordinates extracted with uniform probability. Periodic boundary conditions will be used, since it was shown that, with the same finite number of nodes, they deliver more accurate results with respect to sharp boundaries [9]. Although the real situation is of course three-dimensional, we perform our analysis in two dimensions, restricting ourselves to the case of the average transmitting radius larger than usual building heights. A transmitting radius will be assigned to each of the transmitters. The undirected network arising when transmitting radii are all equal to a given value and two nodes are considered connected if their associated circle overlap, is often in the literature referred to as unit disk graph [2]. We shall go further and also analyze the properties of the directed network constructed by extracting the effective transmission radius of each node from a given probability distribution with finite support. Links in this directed network will be drawn from a transmitter towards all other transmitting devices lying inside its effective covered area. This kind of network would be very close to what we would get in reality.
of a transmitter. In case of different effective transmitting radii \( R_i \), we must have \( T = P_i/R_i^2 \) for each \( R_i \). If we have \( N \) transmitters with ERP \( P_i \) covering an area \( A \), we get for the total radiated power \( P \) the expression \( P = \sum N_i P_i = T \sum N_i R_i^2 \). If the effective transmitting circles were not overlapping, then the area covered by the signal would be \( A = \sum N_i \pi R_i^2 \) and consequently \( P = TA/\pi \), expression that does depend neither on the \( N_i \) nor on the \( R_i \). We stress that this result is true if the transmitting circles do not overlap.

We shall see later in this paper that in the case transmission centers scattered with random uniform probability in the area \( A \), the number of transmitters with effective area \( \sigma = \pi R^2 \) needed to cover the space would be \( N = \frac{A}{\pi \sigma} \log \frac{1}{\rho} \) rather than simply \( N = A/\sigma \). Intuitively we need more transmitters than in the overlap free case to cover a given area, since circles with random extracted centers are not optimized to cover holes. In this case, the total electric power needed to maintain the network would be \( P = \frac{TA}{\pi} \sum \log \frac{1}{\rho} \). Consequently, the minimization of total power may be achieved by using transmitters with large effective transmitting radii, whose transmitting power should take into account both human health and the possibility to assure interference free communications.

### III. UNDIRECTED AD-HOC NETWORKS

Given a certain number of nodes (transmitters) with an associated effective signal area in a unit square with periodic boundary conditions (PBC), the undirected ad-hoc network is constructed by joining two nodes whose signal areas overlap. In case of signal areas with a circular shape of given radius \( R \), a link is drawn whenever two nodes are less than the sum of their signal radii apart.

#### A. Fixed transmitting power

In this section we shall analyze the case of an undirected geometrical network, where nodes have the same transmitting radius \( R \). Two nodes are connected if their mutual distance is less than \( 2R \). Equivalently, two nodes are connected if one of them falls inside a circle of radius \( 2R \) centered on the other. In continuum percolation theory, the area of this circle with double radius is often to as excluded volume \( V_{ex} = 4\pi R^2 \). If we restrict our system in a unit square, the excluded volume represents directly the probability that two random nodes are connected. Since the \( N \) nodes are drawn independently and uniformly at random, the probability distribution of the degree (connectivity) \( k \) will be given by a binomial distribution

\[
\binom{N}{k} p^k (1-p)^{N-k} \tag{1}
\]

with \( p = V_{ex} \), and the average degree will be simply given by \( \alpha = \langle k \rangle = N p = N V_{ex} \). At the thermodynamic limit, when \( N \) grows to infinity while \( \alpha \) remains constant, we can pose questions about the emergence of a macroscopic large cluster, i.e. a cluster with size growing as \( N \). Percolation theory predicts the existence of such cluster and that its appearance is ruled by a phase transition at some critical value of \( \alpha = \alpha_c \). Since there exists no analytical calculation able to devise this critical value yet, \( \alpha_c \) must be computed numerically and its value in 2-dimensions is found to lie around \( \alpha_c = 4.51 \) [9].

If we intend to realize an ad-hoc network with ideal transmitting devices with fixed transmitting radius and we need to ensure the presence of a macroscopic large cluster of nodes, then we have to choose a spatial node density greater than the critical value of \( N_c = \alpha_c/V_{ex} \). What we really like to have is to set up a node spatial density in order to guarantee, with high probability, the emergence of a connected network, where all nodes are connected. The problem of connectivity has already been dealt with in literature and remains an hot topic today as well.

The basic question one poses is: given a certain number \( N \) of transmitters, how should one choose the excluded volume \( V_{ex} \)? In our two-dimensional case the excluded area \( A \) in order to have a totally connected network with high probability? An answer was already given by Gupta and Kumar [11], who showed that if \( N \) nodes are placed in a disk of unit area and each node transmits at a power level so as to cover a circular area

\[
A = \pi R^2 = \log N + c(N) \tag{2}
\]

then the resulting network is asymptotically connected with probability one if and only if \( c(N) \to +\infty \). The term \( c(N) \) is still allowed to grow slower than the logarithm, such that expressions like \( c(N) \approx \log \log N \) are feasible.

Variants of this theorem can also be found in literature. We cite here the work of Xue and Kumar [12], who faced the problem from the point of view of the number of nearest neighbors needed to ensure total connectivity with high probability (they proceed that the number of nearest neighbors should scale as \( O(\log N) \) and conjectured that the multiplicative constant should be strictly one).

**Minimally connected networks with arbitrary shaped agents**

In this section, we shall show with simple probabilistic reasonings, that in the case of a geometrical random network with \( N \) nodes placed uniformly at random in a unit square in two-dimensions and having excluded area \( A \), the network is asymptotically minimally connected if \( N = \log A / \log(1-A) \) or equivalently \( A \approx (\log N - \log \log N)/N \). By “minimally connected” we denote a network with at least two of its nodes connected. The reasoning proceeds as follows.
As previously noticed, if we restrict to a unit square, the probability that a point in the square is covered by a particular node is simply $A$. On the contrary, the complementary probability that a given point in the square is not covered by the area around a chosen node is $(1 - A)$.

It follows immediately that the probability not to cover a given point in the square after $N$ nodes have been placed randomly and independently is $(1 - A)^N$. Since the area of the square is unity, the latter expression coincides with the average free space, i.e. uncovered space, in the square after $N$ nodes have been placed. The network is minimally connected, as average, if the free space is less than the spanned area of the single transmitter, since in this case adding a new node would necessarily lead to an overlap of areas. Thus:

$$\log(1 - A) < A$$

or equivalently

$$N > \frac{\log A}{\log(1 - A)}. \tag{4}$$

Expression (3) may be well approximated as

$$e^{-AN} < A \tag{5}$$

since usually one has $A \ll 1$, i.e. the area spanned by the signal emitted by a transmitter is much less than the required covered territory area [13]. The issue here is to solve the transcendent relation (6) with respect to $A$. To achieve this, we rewrite relation (5) as

$$e^{-AN}/A < 1 \tag{6}$$

and ask which expression for $A(N)$ solves the associated equality. We substitute $A$ with expression (2) and obtain

$$e^{-c(N)}/\log N + c(N) = 1. \tag{7}$$

We observe that by imposing $c(N) = -\log \log N$, we get for the left hand side

$$\log N/\log N - \log \log N \tag{8}$$

expression that goes to one as $N$ grows to infinity. For large $N$ we find then for the area $A(N)$ that ensures a minimally connected network, the approximate expression:

$$A(N) \approx \frac{\log N - \log \log N}{N} \tag{9}$$

We notice that all the previous reasoning do not depend on the shape of the effective transmitting area.

### Connectivity with circular shaped agents

We go back to the case of circular transmitting areas. The probability to have a circular area $S = \pi r^2$ still uncovered after $N$ nodes with area $A = \pi R^2$ have been placed at random in a unit square, is equivalent to the probability not to have any node inside a circle of radius $R + r$, that is $(1 - (\pi(R + r))^2)^N$. If $r = R$ we get

$$(1 - 4\pi R^2)^N \approx e^{-4NA} = e^{-NA_{ex}} \tag{10}$$

with $A_{ex}$ denoting the excluded area as defined above in the text. By substituting expression (2) in the previous relation we get

$$e^{-4NA} < \frac{1}{N^4} \tag{11}$$

that thus gives an estimation of the probability that after $N$ nodes carrying a transmission area $A(N)$ have been placed in the box, a free uncovered circle of area $A(N)$ emerges. In that case, if the next extracted node might fall inside this area and would be disconnected from the others. The complementary probability $(1 - \frac{1}{N^4})$, gives then a pessimistic estimation of the probability that a geometric network constructed as above would display disconnected sections. Finally, If $N \to \infty$ and $A(N)$ is chosen as in Eq. (2) the resulting network is connected with high probability.

Further, the average number of nearest neighbors of a node with transmission radius $R$ in a unit box, i.e. its average connectivity $\langle k \rangle$ was previously mentioned to be equal to the excluded volume or area times $N$. In our case $\langle k \rangle = 4NA(N)$, and by substituting again expression (2) we get

$$\langle k \rangle = 4\log N + 4c(N) > 4\log N. \tag{12}$$

Xue and Kumar demonstrated that if $\langle k \rangle = O(\log N) \approx \gamma \log N$ then the corresponding random geometric network is connected and conjectured that $\gamma$ should be unity [12]. Eq. (12), instead, suggests that the multiplicative constant $\gamma$ should be larger than 4.

### B. Random transmission power

As already discussed in the introduction, the real implementation of an ad-hoc multihop network would be characterized by the presence of nodes with transmitting power drawn from a random distribution density rather than presenting a fixed transmitting radius (Dirac’s delta peaked distribution density). In this section we shall generalize some network characteristic quantities already known for the fixed power case. In the following we shall speak indifferently about radius distribution density $p(R)$ and transmitting area distribution density $P(A)$. In the particular case of circular areas the respective distribution densities are connected by the equiprobability law
\[ P(A)dA = p(R)dR \] of dependent events, i.e.
\[ P(A) = p(R) \frac{dR}{dA} = \frac{p(\sqrt{A/\pi})}{2\sqrt{\pi A}}. \quad (13) \]
The transmitting radius probability densities that we shall use for our analysis will be:

- **R** \text{ fixed radius:} \[ p(R) = \delta(R) \text{ with } R = 0.01; \]
- **2R** \text{ two radii:} \[ p(R) = (\delta(R_1) + \delta(R_2))/2 \text{ with } R_1 = 0.003 \text{ and } R_2 = 0.02; \]
- **unif** \text{ uniform distribution density:} \[ p(R) = \Theta(R - R_1)\Theta(R_2 - R)/((R_2 - R_1)) \text{ with } \Theta(x) \text{ representing the Heaviside theta function with value one if its argument is non negative. } R_1 \text{ and } R_2 \text{ are chosen as above.} \]
- **bpl** \text{ bounded power-law:} \[ p(R) = c\Theta(R - R_1)\Theta(R_2 - R)R^{-2}, \text{ with } c \text{ normalization factor and } R_1 \text{ and } R_2 \text{ chosen as above.} \]

We still shall consider our system inside a unit square with PBC. In Fig. 1 we show a random geometric network generated by the uniform radius distribution density above with 5000 nodes together with its topological representation.

1. **Free space**

The probability to cover a point in the unit box by means of a randomly extracted node with given transmitting area \( A \) is obtained by the product of the probability to extract area \( A \) from its distribution density \( P(A) \) times the probability \( A \) that the point in the box falls in that area: \( AP(A) \). Generally we get, after one node extraction, that the probability to cover a given point is \( \sum_A AP(A) = \langle A \rangle \), where the discrete sum turns into an integral if \( A \) is a continuous variable. We see that now the average transmitting area \( \langle A \rangle \) takes the place of the area \( A \) of the previous fixed transmitting power case of section III A. After \( N \) independent node extractions the average free uncovered total space \( F(N) \) will be
\[ F(N) = (1 - \langle A \rangle)^N \approx e^{-N\langle A \rangle}. \quad (14) \]

As for the case of fixed transmitting radius, it is possible to estimate the minimum number of nodes necessary to have a connected network once the distribution density of areas is fixed. In fact relation (3) now becomes
\[ (1 - \langle A \rangle)^N < A_0 \quad (15) \]
where \( A_0 \) is the minimum value of the transmitting areas. Thus, the network is connected when \( N > \log A_0/\log(1 - \langle A \rangle) \).

2. **Degree distribution**

The probability that a the circle of radius \( r \) surrounding a given node, will intersect a circle of radius \( R \) associated to another node is equal to the area of an effective circle of radius \( r + R \) (sort of effective r-dependent excluded area) times the probability to get the node with radius \( R \). The average value of the degree of the node with radius \( r \) will be:
\[ k(r) = N\pi \int (r + R)^2p(R)dR. \quad (16) \]

The previous equation defines also the degree distribution \( P(k) \), by using the analogous of formula (13), i.e.
\[ P(k) = p(r(k))r'(k). \]
The average value of the degree in the network will be
\[ \langle k \rangle = N\pi \int (r + R)^2p(r)dR \, dr = 2\pi N[(r^2) + \langle r^2 \rangle]. \quad (17) \]
Given a node in the network, its clustering coefficient is calculated as the fraction of all triangles formed by the node itself and its nearest neighbors, normalized to all possible triangle so defined. In a communication frame, the clustering coefficient might be of some importance, since if nodes a, b and c are connected, then the interruption of the direct communication between a and b would not preclude the their mutual communication, as the information would pass through c. This idea should be developed in the near future. The clustering coefficient of a network is defined as the average clustering coefficient of its nodes. Fig. 4 depicts the total clustering coefficient of networks generated as above.

IV. DIRECTED AD-HOC NETWORKS

The undirected ad-hoc network as described in the previous sections are an ideal abstraction of what an effective working complex communicating network should be. It seems more reasonable to define a connection from node a to node b if the effective area of signal emitted by a embraces node b. In this way a more realistic network is defined directed. A next complication, which we shall not deal with here, is to assign to each directed link a weight, possibly modeling the different bandwidth capabilities of nodes. For this latter case we already developed in the past year activity of WP2.2.1 the mathematical tools to analyze the relevant statistical properties of the network. The case of all node of the network with fixed transmitting power, can be easily generalized in the frame of directed networks, by simply replacing the excluded area with $\pi R^2$. In fact if two nodes are connected then the link must be reciprocal, while these two nodes must lie one in the transmitting effective area of the other. Differences from a trivial undirected network will arise when nodes may posses different transmitting radii.

A. Degree distributions

In a directed network there are two quantities related to the connectivity of a node: the in-degree (the number of incoming links) and the out-degree (number of outgoing links). Given a node $m$ with associated transmitting area $A_m$, the probability to draw a link towards another node is simply $A_m$ (we still restrict to a unit square). The out-degree distribution of node $m$ is then given by the binomial distribution of Eq. 1 with $k = k_{out,m}$ and $p = A_m$. The average $k_{out,m}$ is $\langle k_{out,m} \rangle = N A_m$ so that the the out-degree distribution $p(k_{out})$ will be essentially given by the distribution of the transmitting areas. The average out-degree is simply $\langle k_{out} \rangle = N \langle A \rangle$. The in-degree obeys the same relations.
B. Percolative phase transition

The questions about the existence of a percolative phase transition should be now addressed to the emergence of a macroscopic strongly connected component (SCC), defined to be a portion of graph whose nodes can be reached from all others. We shall carry this analysis in the nearest future.

V. ANALYSIS SOFTWARE

We developed an open source software with which we are able to study many of the quantities dealt with in this report. It has the possibility to generates networks with given criteria ranging from the preferential attachment rule to the geometric random networks. It automatically performs ensemble averages, by specifying the number of system copies to be handled. This software can be downloaded from the web page http://pil.phys.uniroma1.it/~servedio/Software.html.

[13] We also carried on our analysis without this approximation and got same results. By using expression (5) instead of expression (3) we simplify the analytical treatment of the problem.