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Abstract

In this work we give precise asymptotic expressions on the probability of the existence of fixed-size components at the threshold of connectivity for random geometric graphs.

1 Introduction and basic results on Random Geometric Graphs.

Recently, quite a bit of work has been done on Random Geometric graphs, due to the importance of these graphs as theoretical models for ad hoc networks (for applications we refer to [2]). Most of the theoretical results on random geometric graphs can be found in the book by M. D. Penrose [3]. In this section we succinctly recall the results needed to motivate and prove our main theorem.

Given a set of \(n\) vertices and a positive real \(r = r(n)\), each vertex is placed at some random position in the unit torus \([0, 1)\)^2 selected independently and uniformly at random (u.a.r.). We denote by \(X_i = (x_i, y_i)\) the random position of agent \(i\) for \(i \in \{1, \ldots, n\}\), and let \(\mathcal{X} = \mathcal{X}(n) = \bigcup_{i=1}^{n} X_i\). Note that with probability 1, no two vertices choose the same position and thus we restrict the attention to the case that \(|\mathcal{X}| = n\). We define \(G(\mathcal{X}; r)\) as the random graph having \(\mathcal{X}\) as the vertex set, and with an edge connecting each pair of vertices \(X_i\) and \(X_j\) in \(\mathcal{X}\) at distance \(d(X_i, X_j) \leq r\), where \(d(\cdot, \cdot)\) denotes the Euclidean distance in the torus.

Unless otherwise stated, all our stated results are asymptotic as \(n \to \infty\). As usual, the abbreviation a.a.s. stands for asymptotically almost surely, i.e. with probability \(1 - o(1)\). All logarithms in this paper are natural logarithms.

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Let $K_1$ be the random variable counting the number of isolated vertices in $G(\mathcal{X}; r)$. Then, by multiplying the probability that one vertex is isolated by the number of vertices we obtain

$$E(K_1) = n(1 - \pi r^2)^{n-1} = ne^{-\pi r^2 n - O(r^4 n)}. \quad (1)$$

Define $\mu := ne^{-\pi r^2 n}$. Observe from the previous expression that $\mu$ is closely related to $E(K_1)$. In fact, $\mu = o(1)$ iff $E(K_1) = o(1)$, and if $\mu = \Omega(1)$ then $E(K_1) \sim \mu$. Moreover the asymptotic behaviour of $\mu$ characterizes the connectivity of $G(\mathcal{X}; r)$. The connectivity behaviour can be summarized as follows (see [3] for a survey of the work on the connectivity threshold for random geometric graphs)

- If $\mu \to 0$, then a.a.s. $G(\mathcal{X}; r)$ is connected.
- If $\mu = \Theta(1)$, then a.a.s. $G(\mathcal{X}; r)$ consists of one giant component of size $> n/2$ and a Poisson number (with parameter $\mu$) of isolated vertices.
- If $\mu \to \infty$, then a.a.s. $G(\mathcal{X}; r)$ is disconnected.

From the definition of $\mu$ we deduce that $\mu = \Theta(1)$ iff $r = \sqrt{\frac{\ln n + O(1)}{\pi n}}$. Therefore as a weaker consequence we conclude that the property of connectivity of $G(\mathcal{X}; r)$ exhibits a sharp threshold at $r_c := \sqrt{\frac{\ln n}{\pi n}}$. Note that the previous classification of the connectivity of $G(\mathcal{X}; r)$, indicates that if $\mu = \Theta(1)$, the components of size 1 are predominant and those components have the main contribution to the connectivity of $G(\mathcal{X}; r)$. In fact if $\mu = \Theta(1)$, the probability that $G(\mathcal{X}; r)$ has some component of size greater than 1 other than the giant component is $o(1)$.

On the other hand, M.D. Penrose [3] studied the number of components in $G(\mathcal{X}; r)$, which are isomorphic to a given fixed graph; equivalently, he studied the probability of finding components of a given size in $G(\mathcal{X}; r)$. However the range of radii $r$ covered by Penrose does not exceed the thermodynamical threshold $\Theta(\sqrt{1/n})$ where a giant component appears at $G(\mathcal{X}; r)$, which is below the connectivity threshold treated in the present paper. In fact, a percolation argument in [3] only shows that with probability $1 - o(1)$ no components other than isolated vertices and the giant one exist at the connectivity threshold, whithout giving accurate bounds on this probability.

In the remaining of the paper, we prove that at $r_c$, given a fixed $\ell > 1$, the probability of having components of size exactly $\ell$ is $\Theta\left(\frac{1}{\log^{-1} n}\right)$. Moreover, in the process of the proof we characterize the different types of components that could exist at the connectivity threshold $r_c$.

2 Mid-size components

Given a component $\Gamma$ of $G(\mathcal{X}; r)$, $\Gamma$ is embeddable if it can be mapped into the square $[r, 1 - r]^2$ by a translation in the torus. Embeddable components do not wrap around the torus.
Components which are not embeddable must have a large size (at least $\Omega(1/r)$). Sometimes several non-embeddable components can coexist together (see Figure 1). However, there are some non-embeddable components which are so spread around the torus that do not allow any room for other non-embeddable ones. Call these components solitary. Clearly, by definition we can have at most one solitary component. We cannot disprove the existence of a solitary component, since with probability $1 - o(1)$ there exists a giant component of this nature. For components which are not solitary, we give asymptotic bounds on the probability of their existence according to their size.

Given a fixed integer $\ell \geq 1$, let $K_\ell$ be the number of components in $G(\mathcal{X}; r)$ of size exactly $\ell$. For large enough $n$, we can assume these to be embeddable, since $r = o(1)$. Moreover, for any fixed $\epsilon > 0$, let $K'_{\epsilon, \ell}$ be the number of components of size exactly $\ell$ which have all their vertices at distance at most $\epsilon r$ from their leftmost one. Finally, $\tilde{K}_\ell$ denotes the number of components of size at least $\ell$ and which are not solitary. In Figure 2 an example of a component of size exactly $\ell = 9$ is given, which has all its vertices at distance at most $\epsilon r$ from the leftmost one $u$.

Notice that $K'_{\epsilon, \ell} \leq K_\ell \leq \tilde{K}_\ell$. However, in the following we show that asymptotically all the weight in the probability that $\tilde{K}_\ell > 0$ comes from components which
also contribute to $K'_{\epsilon, \ell}$ for $\epsilon$ arbitrarily small. This means that the more common
components of size at least $\ell$ are cliques of size exactly $\ell$ with all their vertices close
together.

**Lemma 1.** Let $\ell \geq 2$ be a fixed integer, and $0 < \epsilon < 1/2$ be also fixed. Assume that
$\mu = \Theta(1)$. Then,

$$E(K'_{\epsilon, \ell}) = \Theta(1/\log^{\ell-1} n)$$

**Proof.** First observe that with probability 1, for each component $\Gamma$ which contributes
to $K'_{\epsilon, \ell}$, $\Gamma$ has a unique leftmost vertex $X_i$ and the vertex $X_j$ in $\Gamma$ at greatest distance
from $X_i$ is also unique. Hence, we can restrict our attention to this case.

Fix an arbitrary set of indices $J \subset \{1, \ldots, n\}$ of size $|J| = \ell$, with two distinguished
elements $i$ and $j$. Denote by $S = \bigcup_{k \in J} X_k$ the set of random points in $\mathcal{X}$ with indices
in $J$. Let $\mathcal{E}$ be the following event: All vertices in $\mathcal{Y}$ are at distance at most $\epsilon r$ from
$X_i$ and to the right of $X_i$; vertex $X_j$ is the one in $\mathcal{Y}$ with greatest distance from $X_i$; and the vertices of $\mathcal{Y}$ form a component of $G(\mathcal{X}; r)$. If $\Pr(\mathcal{E})$ is multiplied by the
number of possible choices of $i, j$ and the remaining $\ell - 2$ elements of $J$, we get

$$E K'_{\epsilon, \ell} = n(n - 1) \left(\frac{n - 2}{\ell - 2}\right) \Pr(\mathcal{E}). \quad (1)$$

In order to bound the probability of $\mathcal{E}$ we need some definitions. Let $\rho = d(X_i, X_j)$
and let $\mathcal{S}$ be the set of all points in the torus $[0, 1]^2$ which are at distance at most $r$
from some vertex in $\mathcal{Y}$ (see Figure 3 for an example). (Notice that $\rho$ and $\mathcal{S}$ depend
on the set of random points $\mathcal{Y}$).

We first need bounds of $\text{Area}(\mathcal{S})$ in terms of $\rho$. Observe that $\mathcal{S}$ is contained in the
circle of radius $r + \rho$ and center $X_i$, and then

$$\text{Area}(\mathcal{S}) \leq \pi(r + \rho)^2. \quad (2)$$

Now let $i_L = i, i_R, i_T$ and $i_B$ be respectively the indices of the leftmost, rightmost,
topmost and bottommost vertices in $\mathcal{Y}$ (some of these indices possibly equal). Assume
w.l.o.g. that the vertical length of $\mathcal{Y}$ (i.e. the vertical distance between $X_{i_L}$ and $X_{i_R}$)
is at least $\rho/\sqrt{2}$. Otherwise, the horizontal length of $\mathcal{Y}$ has this property and we can
rotate the descriptions in the argument. The upper halfcircle with center \(X_{i_T}\) and the lower halfcircle with center \(X_{i_B}\) are disjoint and are contained in \(S\). If \(X_{i_R}\) is at greater vertical distance from \(X_{i_T}\) than from \(X_{i_B}\), then consider the rectangle of height \(\rho/(2\sqrt{2})\) and width \(r - \rho/(2\sqrt{2})\) with one corner on \(X_{i_R}\) and above and to the right of \(X_{i_R}\). Otherwise, consider the same rectangle below and to the right of \(X_{i_R}\). This rectangle is also contained in \(S\) and its interior does not intersect the previously described halfcircles. Analogously, we can find another rectangle of height \(\rho/(2\sqrt{2})\) and width \(r - \rho/(2\sqrt{2})\) to the left of \(X_{i_L}\) and either above or below \(X_{i_L}\) with the same properties. Hence,

\[
\text{Area}(S) \geq \pi r^2 + 2 \left( \frac{\rho}{2\sqrt{2}} \right) \left( r - \frac{\rho}{2\sqrt{2}} \right).
\]

From (2), (3) and the fact that \(\rho < r/2\), we can write

\[
\pi r^2 \left( 1 + \frac{\rho}{6r} \right) < \text{Area}(S) < \pi r^2 \left( 1 + \frac{5\rho}{2r} \right) < \frac{9\pi}{4} r^2.
\]

Now consider the probability \(P\) that the \(n - \ell\) vertices not in \(Y\) lie outside \(S\). Clearly \(P = (1 - \text{Area}(S))^{n-\ell}\). Moreover, by (4) and using the fact that \(e^{-x} \leq 1 - x \leq e^{-x}\) for all \(x \in [0, 1/2]\), we obtain

\[
e^{-\frac{1+5\rho/(2r)}{\pi r^2}n-(9\pi r^2/4)^2n} < P < e^{-\frac{(1+\rho/(6r))\pi r^2n}{(1-9\pi r^2/4)^\ell}},
\]

and after a few manipulations

\[
\left( \frac{\mu}{n} \right)^{1+5\rho/(2r)} e^{-(9\pi r^2/4)^2n} < P < \left( \frac{\mu}{n} \right)^{1+\rho/(6r)} \frac{1}{(1-9\pi r^2/4)^\ell}.
\]

Event \(E\) can also be described as follows: There is some nonnegative real \(\rho \leq \epsilon r\) such that \(X_j\) is placed at distance \(\rho\) from \(X_i\) and to the right of \(X_i\); all the remaining vertices in \(Y\) are inside the halfcircle of center \(X_i\) and radius \(\rho\); and the \(n - \ell\) vertices not in \(Y\) lie outside \(S\). Hence, \(\text{Pr}(E)\) can be bounded from above (below) by integrating with respect to \(\rho\) the probability density function of \(d(X_i, X_j)\) times the probability that the remaining \(\ell - 2\) selected vertices lie inside the right halfcircle of center \(X_i\) and radius \(\rho\) times the upper (lower) bound on \(P\) we obtained in (5):

\[
\Theta(1) I(5/2) \leq \text{Pr}(E) \leq \Theta(1) I(1/6),
\]

where

\[
I(\beta) = \int_0^{\epsilon r} \pi \rho \left( \frac{\pi}{2} \rho^2 \right)^{\ell-2} \frac{1}{n^{1+\beta\rho/r}} d\rho.
\]

\[
= \frac{2}{n} \left( \frac{\pi}{2} r^2 \right)^{\ell-1} \int_0^{\epsilon} x^{2\ell-3} n^{-\beta x} dx
\]
Since $\ell$ is fixed, for $\beta = 5/2$ or $\beta = 1/6$,

$$I(\beta) = \Theta \left( \frac{\log^{\ell-1} n}{n^\ell} \right) \int_0^\epsilon x^{2\ell-3} n^{-\beta x} \, dx$$

$$= \Theta \left( \frac{\log^{\ell-1} n}{n^\ell} \right) \frac{(2\ell - 3)!}{(\beta \log n)^{2\ell-2}}$$

$$= \Theta \left( \frac{1}{n^\ell \log^{\ell-1} n} \right), \quad (8)$$

and the statement follows from (1), (6) and (8).

\[ \square \]

**Lemma 2.** Let $\ell \geq 2$ be a fixed integer. Let $\epsilon > 0$ be also fixed. Assume that $\mu = \Theta(1)$. Then

$$\Pr \left[ \tilde{K}_\ell - K'_{\epsilon,\ell} > 0 \right] = O(1/\log^\ell n)$$

**Proof.** We assume throughout this proof that $\epsilon \leq 10^{-18}$, and prove the claim for this case. The case $\epsilon > 10^{-18}$ follows from the fact that $(\tilde{K}_\ell - K'_{\epsilon,\ell}) \leq (\tilde{K}_\ell - K'_{10^{-18},\ell})$.

Consider all the possible components in $G(\mathcal{X}; r)$ which are not solitary. Remove from these components the ones of size at most $\ell$ and diameter at most $\epsilon r$, and denote by $M$ the number of remaining components. By construction $\tilde{K}_\ell - K'_{\epsilon,\ell} \leq M$, and therefore it is sufficient to prove that $\Pr(M > 0) = O(1/\log^\ell n)$. The components counted by $M$ are classified into several types according to their size and diameter. We deal with each type separately.

**Part 1.** Consider all the possible components in $G(\mathcal{X}; r)$ which have diameter at most $\epsilon r$ and size between $\ell + 1$ and $\log n/37$. Call them components of type 1, and let $M_1$ denote their number.

For each $k$, $\ell + 1 \leq k \leq \log n/37$, let $E_k$ be the expected number of components of type 1 and size $k$. We observe that these components have all of their vertices at distance at most $\epsilon r$ from the leftmost one. Therefore, we can apply the same argument we used for bounding $E K'_{\epsilon,\ell}$ in the proof of Lemma 1. Note that (1), (6) and (7) are also valid for sizes not fixed but depending on $n$. Thus we obtain

$$E_k \leq O(1)n(n-1) \binom{n-2}{k-2} I(1/6),$$

where $I(1/6)$ is defined in (7). We use the fact that $\binom{n-2}{k-2} \leq (\frac{ne}{k-2})^{k-2}$ and get

$$E_k = O(1) \log n \left( \frac{e \log n}{2k-2} \right)^{k-2} \int_0^\epsilon x^{2k-3} n^{-x/6} \, dx \quad (9)$$

The expression $x^{2k-3} n^{-x/6}$ can be maximized for $x \in \mathbb{R}^+$ by elementary techniques, and we deduce that

$$x^{2k-3} n^{-x/6} \leq \left( \frac{2k-3}{(e/6) \log n} \right)^{2k-3}.$$
Figure 4: The tessellation for counting components of type 2 with two particular boxes marked.

Then we can bound the integral in (9) and get

\[ E_k = O(1) \log n \left( \frac{e \log n}{2k - 2} \right)^{k-2} \epsilon \left( \frac{2k - 3}{(e/6) \log n} \right)^{2k-3} \]

\[ = O(1) \left( \frac{36}{2e (k - 2) \log n} \right)^{k-2} k \]

Note that for \( k \leq \log n/37 \) the expression \( \left( \frac{36}{2e (k - 2) \log n} \right)^{k-2} k \) is decreasing with \( k \). Hence we can write

\[ E_k = O \left( \frac{1}{\log^{\ell+1} n} \right), \quad \forall k : \ell + 3 \leq k \leq \frac{1}{37} \log n. \]

Moreover the bounds \( E_{\ell+1} = O(1/\log^\ell n) \) and \( E_{\ell+2} = O(1/\log^{\ell+1} n) \) are obtained from Lemma 1, and hence

\[ EM_1 = \sum_{k=\ell+1}^{37 \log n} E_k = O \left( \frac{1}{\log^\ell n} \right) + O \left( \frac{1}{\log^{\ell+1} n} \right) + \frac{\log n}{37} O \left( \frac{1}{\log^{\ell+1} n} \right) = O \left( \frac{1}{\log^\ell n} \right), \]

and then \( \Pr(M_1 > 0) \leq EM_1 = O(1/\log^\ell n) \).

Part 2. Consider all the possible components in \( G(X; r) \) which have diameter at most \( \epsilon r \) and size greater than \( \log n/37 \). Call them components of type 2, and let \( M_2 \) denote their number.

We tessellate the torus with square cells of side \( y = \lfloor (\epsilon r)^{-1} \rfloor^{-1} \) (\( y \geq \epsilon r \) but also \( y \sim \epsilon r \)). We define a box to be a square of side \( 2y \) consisting of the union of 4 cells of the tessellation. Consider the set of all possible boxes. Note that any component of type 2 must be fully contained in some box (see Figure 4).

Let us fix a box \( b \). Let \( W \) be the number of vertices which are deployed inside \( b \). Clearly \( W \) has a binomial distribution with mean \( EW = (2y)^2 n \sim (2\epsilon^2)^2 \log n / \pi \). By setting \( \delta = \log n / 37 EW \) and applying Chernoff inequality to \( W \), we have

\[ \Pr(W > \frac{1}{37} \log n) = \Pr(W > (1 + \delta)EW) \leq \left( \frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^{EW} = n^{-\frac{(\log(1+\delta) - \frac{\delta}{1+\delta})}{37}}. \]
Note that $\delta \sim \frac{\pi}{148\epsilon^2} - 1 > e^{79}$, and therefore

$$\Pr(W > \frac{1}{37} \log n) < n^{-2.1}.$$ 

Then taking a union bound over the set of all $\Theta(r^{-1}) = \Theta(n/\log n)$ boxes, the probability that there is some box with more than $\frac{1}{37} \log n$ vertices is $O(1/(n^{1.1} \log n))$. Then since each component of type 2 is contained in some box, we have

$$\Pr(M_2 > 0) = O(1/(n^{1.1} \log n)).$$

**Part 3.** Consider all the possible components in $G(X; r)$ which are embeddable and have diameter at least $\epsilon r$. Call them components of type 3, and let $M_3$ denote their number.

We tessellate the torus into square cells of side $\alpha r$, for some $\alpha = \alpha(\epsilon) > 0$ fixed but sufficiently small. Let $\Gamma$ be a component of type 3. Let $S = S_\Gamma$ be the set of all points in the torus $[0, 1)^2$ which are at distance at most $r$ from some vertex in $\Gamma$. Remove from $S$ the vertices of $\Gamma$ and the edges (represented by straight segments) and denote by $S'$ the outer connected topologic component of the remaining set. By construction, $S'$ must contain no vertex in $X$ (see Figure 5, left picture).

Now let $i_L, i_R, i_T$ and $i_B$ be respectively the indices of the leftmost, rightmost, topmost and bottommost vertices in $\Gamma$ (some of these indices possibly equal). Assume w.l.o.g. that the vertical length of $\Gamma$ (i.e. the vertical distance between $X_{i_T}$ and $X_{i_B}$) is at least $\epsilon r/\sqrt{2}$. Otherwise, the horizontal length of $\Gamma$ has this property and we can rotate the descriptions in the argument. The upper halfcircle with center $X_{i_T}$ and the lower halfcircle with center $X_{i_B}$ are disjoint and are contained in $S'$. If $X_{i_B}$ is at greater vertical distance from $X_{i_T}$ than from $X_{i_B}$, then consider the rectangle of height $\epsilon r/(2\sqrt{2})$ and width $r - \epsilon r/(2\sqrt{2})$ with one corner on $X_{i_B}$ and above and to the right of $X_{i_R}$. Otherwise, consider the same rectangle below and to the right of $X_{i_B}$. This rectangle is also contained in $S'$ and its interior does not intersect the previously described halfcircles. Analogously, we can find another rectangle of height $\epsilon r/(2\sqrt{2})$ and width $r - \epsilon r/(2\sqrt{2})$ to the left of $X_{i_L}$ and either above or below $X_{i_L}$ with the same properties. Hence, taking into account that $\epsilon \leq 10^{-18}$, we have

$$\text{Area}(S') \geq \pi r^2 + 2 \left( \frac{\epsilon r}{2\sqrt{2}} \right) \left( r - \frac{\epsilon r}{2\sqrt{2}} \right) > \left( 1 + \frac{\epsilon}{5} \right) \pi r^2. \quad (10)$$
Let $S^*$ be the union of all the cells in the tessellation which are fully contained in $S'$. We lose a bit of area compared to $S'$. However, if $\alpha$ was chosen small enough, we can guarantee that $S^*$ is topologically connected and has area $\text{Area}(S^*) \geq (1 + \epsilon/6)\pi r^2$. This $\alpha$ can be chosen to be the same for all components of type 3 (see Figure 5, right picture).

Hence, we showed that the event $(M_3 > 0)$ implies that some connected union of cells $S^*$ of area $\text{Area}(S^*) \geq (1 + \epsilon/6)\pi r^2$ contains no vertices. By removing some cells from $S^*$, we can assume that $(1 + \epsilon/6)\pi r^2 \leq \text{Area}(S^*) < (1 + \epsilon/6)\pi r^2 + \alpha^2 r^2$. Let $S^*$ be any union of cells with these properties. (Note that there are $\Theta(1/\alpha^2) = \Theta(n/\log n)$ many possible choices for $S^*$.) The probability that $S^*$ contains no vertices is 

$$(1 - \text{Area}(S^*))^n \leq e^{-(1+\epsilon/6)\pi r^2 n} = \left(\frac{\mu}{n}\right)^{1+\epsilon/6}.$$ 

Therefore, we can take the union bound over all the $\Theta(n/\log n)$ possible $S^*$, and obtain an upper bound of the probability that there is some component of the type 3: 

$$\Pr(M_3 > 0) \leq \Theta\left(\frac{n}{\log n}\right)\left(\frac{\mu}{n}\right)^{1+\epsilon/6} = \Theta\left(\frac{1}{n^{\epsilon/6}\log n}\right).$$ 

**Part 4.** Consider all the possible components in $G(\mathcal{X}; r)$ which are not embeddable but not solitary either. Call them components of type 4, and let $M_4$ denote their number.

We tessellate the torus $[0,1]^2$ into $\Theta(n/\log n)$ small square cells of side length $\alpha r$, where $\alpha > 0$ is a sufficiently small positive constant.

Let $\Gamma$ be a component of type 4. Let $S = S_\Gamma$ be the set of all points in the torus $[0,1]^2$ which are at distance at most $r$ from some vertex in $\Gamma$. Remove from $S$ the vertices of $\Gamma$ and the edges (represented by straight segments) and denote by $S'$ the remaining set. By construction, $S'$ must contain no vertex in $\mathcal{X}$.

Suppose there is a horizontal or a vertical band of width $2r$ in $[0,1]^2$ which does not intersect the component $\Gamma$ (assume w.l.o.g. that it is the topmost horizontal band consisting of all points with the $y$-coordinate in $[1-2r, 1]$). Let us divide the torus into vertical bands of width $2r$. All of them must contain at least one vertex of $\Gamma$, since otherwise $\Gamma$ would be embeddable. Select any 9 consecutive vertical bands and pick one vertex of $\Gamma$ with maximal $y$-coordinate in each one. For each one of these 9 vertices, we select the left upper quartercircle centered at the vertex if the vertex is closer to the right side of the band or the right upper quartercircle otherwise. These nine quartercircles we chose are disjoint and must contain no vertices by construction. Moreover, they belong to the same connected component of the set $S''$, which we denote by $S''$, and which has an area of $\text{Area}(S'') \geq (9/4)\pi r^2$. Let $S^*$ be the union of all the cells in the tessellation of the torus which are completely contained in $S''$. We lose a bit of area compared to $S''$. However, as usual, by choosing $\alpha$ small enough we can guarantee that $S^*$ is connected and it has an area of $\text{Area}(S^*) \geq (11/5)\pi r^2$. Note that this $\alpha$ can be chosen to be the same for all components $\Gamma$ of this kind.

Suppose otherwise that all horizontal and vertical bands of width $2r$ in $[0,1]^2$ contain at least one vertex of $\Gamma$. Since $\Gamma$ is not solitary it must be possible that it
coexists with some other non-embeddable component $\Gamma'$. Then all vertical bands or all horizontal bands of width $2r$ must also contain some vertex of $\Gamma'$ (assume w.l.o.g. the vertical bands do). Let us divide the torus into vertical bands of width $2r$. We can find a simple path $\Pi$ with vertices in $\Gamma'$ which passes through $11$ consecutive bands. For each one of the $9$ internal bands, pick the uppermost vertex of $\Gamma$ in the band below $\Pi$ (in the torus sense). As before each one of these vertices contributes with a disjoint quartercircle which must be empty of vertices, and by the same argument we obtain a connected union of cells of the tessellation, which we denote by $S^*$, with $\text{Area}(S^*) \geq (11/5)\pi r^2$ and containing no vertices.

Hence, we showed that the event $(M_4 > 0)$ implies that some connected union of cells $S^*$ with $\text{Area}(S^*) \geq (11/5)\pi r^2$ contains no vertices. By repeating the same argument we used for components of type $3$ but replacing $(1 + \epsilon/6)\pi r^2$ by $(11/5)\pi r^2$, we get

$$\Pr(M_4 > 0) = \Theta\left(\frac{1}{n^{0.5} \log n}\right).$$

\[ \square \]

**Lemma 3.** Let $\ell \geq 2$ be a fixed integer. Let $0 < \epsilon < 1/2$ be fixed. Assume that $\mu = \Theta(1)$. Then

$$\mathbf{E}\left(K_{\ell}^*\right) \leq O(1/\log^{2\ell-2} n)$$

**Proof.** As in the proof of Lemma 1, we assume that each component $\Gamma$ which contributes to $K_{\ell}^*$ has a unique leftmost vertex $X_i$, and the vertex $X_j$ in $\Gamma$ at greatest distance from $X_i$ is also unique. In fact, this happens with probability $1$.

Choose any two disjoint subsets of $\{1, \ldots, n\}$ of size $\ell$ each, namely $J_1$ and $J_2$, with four distinguished elements $i_1, j_1 \in J_1$ and $i_2, j_2 \in J_2$. For $k \in \{1, 2\}$, denote by $\mathcal{Y}_k = \bigcup_{i \in J_k} X_i$ the set of random points in $\mathcal{X}$ with indices in $J_k$. Let $\mathcal{E}$ be the event that the following conditions hold for both $k = 1$ and $k = 2$: All vertices in $\mathcal{Y}_k$ are at distance at most $\epsilon r$ from $X_{i_k}$ and to the right of $X_{i_k}$; vertex $X_{j_k}$ is the one in $\mathcal{Y}_k$ with greatest distance from $X_{i_k}$; and the vertices of $\mathcal{Y}_k$ form a component of $G(\mathcal{X}; r)$.

If $\Pr(\mathcal{E})$ is multiplied by the number of possible choices of $i_k, j_k$ and the remaining vertices of $J_k$, we get

$$\mathbf{E}[K_{\ell}^*] = O(n^{2\ell})\Pr(\mathcal{E}). \quad (11)$$

In order to bound the probability of $\mathcal{E}$ we need some definitions. For each $k \in \{1, 2\}$, let $\rho_k = d(X_{i_k}, X_{j_k})$ and let $S_k$ be the set of all the points in the torus $[0, 1)^2$ which are at distance at most $r$ from some vertex in $\mathcal{Y}_k$. (Obviously $\rho_k$ and $S_k$ depend on the set of random points $\mathcal{Y}_k$. ) Also define $\mathcal{S} = S_1 \cup S_2$.

Let $\mathcal{F}$ be the event that $d(X_{i_1}, X_{i_2}) > 3r$. This holds with probability $1 - O(r^2)$. In order to bound $\Pr(\mathcal{E} \mid \mathcal{F})$, we apply a similar approach to the one in the proof of Lemma 1. In fact, observe that if $\mathcal{F}$ holds then $S_1 \cap S_2 = \emptyset$. Therefore in view of (4) we can write

$$\pi r^2 (2 + (\rho_1 + \rho_2)/(6r)) < \text{Area}(\mathcal{S}) < \frac{18\pi}{4} r^2. \quad (12)$$
and using the same elementary techniques that gave us \((5)\) we get
\[
(1 - \text{Area}(S))^{n-2\ell} < \left(\frac{\mu}{n}\right)^{2+(\rho_1+\rho_2)/(6r)} \frac{1}{(1 - 18\pi r^2/4)^{2\ell}}.
\]
(13)

Now observe that \(E\) can also be described as follows: For each \(k \in \{1, 2\}\) there is some nonnegative real \(\rho_k \leq \epsilon r\) such that \(X_{1_k}\) is placed at distance \(\rho_k\) from \(X_{1_k}\) and to the right of \(X_{1_k}\); all the remaining vertices in \(Y_k\) are inside the halfcircle of center \(X_{1_k}\) and radius \(\rho_k\); and the \(n - \ell\) vertices not in \(Y_k\) lie outside \(S_k\). In fact, rather than this last condition we only require for our bound that all vertices in \(X \setminus (Y_1 \cup Y_2)\) are placed outside \(S\). Clearly, this has probability \((1 - \text{Area}(S))^{n-2\ell}\). Then, from (13) and following an analogous argument to the one that leads to (6), we obtain the bound
\[
\Pr(E \mid F) \leq \Theta(1) \int_0^{\epsilon r} \int_0^{\epsilon r} \pi \rho_1 \left(\frac{\pi \rho_1}{2}\right)^{\ell-2} \pi \rho_2 \left(\frac{\pi \rho_2}{2}\right)^{\ell-2} \frac{1}{n^{2+(\rho_1+\rho_2)/(6r)}} d\rho_1 d\rho_2
= \Theta(1) I(1/6)^2,
\]
where \(I(1/6)\) is defined in (7). Thus from (8) we conclude
\[
\Pr(E \cap F) \leq \Theta(1) P(F) I(1/6)^2 = O\left(\frac{1}{n^{2\ell \log^{2\ell-2} n}}\right).
\]
(14)

Otherwise, suppose that \(F\) does not hold (i.e. \(d(X_{1_1}, X_{1_2}) \leq 3r\)). Observe that \(E\) implies that \(d(X_{1_1}, X_{1_2}) > r\), since \(X_{1_1}\) and \(X_{1_2}\) must belong to different components. Hence the circles with centers on \(X_{1_1}\) and \(X_{1_2}\) and radius \(r\) have an intersection of area less than \((\pi/2)r^2\). These two circles are contained in \(S\) and then we can write \(\text{Area}(S) \geq (3/2)\pi r^2\). Note that \(E\) implies that all vertices in \(X \setminus (Y_1 \cup Y_2)\) are placed outside \(S\) and that for each \(k \in \{1, 2\}\) all the vertices in \(Y_k \setminus \{X_{1_k}\}\) are at distance at most \(\epsilon r\) and to the right of \(X_{1_k}\). This gives us the following rough bound
\[
\Pr(E \mid \overline{F}) \leq \left(\frac{\pi}{2}(\epsilon r)^2\right)^{2\ell-2} \left(1 - \frac{3\pi}{2} r^2\right)^{n-2\ell} = O(1) \left(\frac{\log n}{n}\right)^{2\ell-2} \left(\frac{\mu}{n}\right)^{3/2}.
\]

Multiplying this by \(\Pr(F) = O(r^2) = O(\log n/n)\) we obtain
\[
\Pr(E \cap \overline{F}) = O\left(\frac{\log^{2\ell-1} n}{n^{2\ell+1/2}}\right),
\]
(15)
which is negligible compared to (14). The statement follows from (11), (14) and (15).

Our main theorem now follows easily.

**Theorem 4.** Let \(\ell \geq 2\) be a fixed integer. Let \(0 < \epsilon < 1/2\) be fixed. Assume that \(\mu = \Theta(1)\). Then
\[
\Pr\left[\tilde{K}_\ell > 0\right] \sim \Pr[K_\ell > 0] \sim \Pr[K'_{r,\ell} > 0] = \Theta\left(\frac{1}{\log^{\ell-1} n}\right).
\]
Proof. From Corollary 1.12 in [1], we have

\[ EK_{\epsilon,\ell}' - \frac{1}{2}E[K_{\epsilon,\ell}'^2] \leq \Pr(K_{\epsilon,\ell}' > 0) \leq EK_{\epsilon,\ell}', \]

and therefore by Lemmata 1 and 3 we obtain

\[ \Pr(K_{\epsilon,\ell}' > 0) = \Theta(1/\log^{\ell-1} n). \]

Combining this and Lemma 2, yields the statement. \qed

References

